Integration by Parts

$$\int \boldsymbol{u\,dv = uv - \int v\ du}$$

$$\int_a^b \boldsymbol{u\,dv = [uv]_a^b - \int_a^b v\,du}$$

1. Repeated Use

a) .

$$\int_1^e x^3\ \ln x\ dx\ ;\ \text{Let: } u=\ln x\,, du=\frac{dx}{x};\ \ dv=x^3\,dx, v=\frac{x^4}{4}$$

$$\therefore \int_1^e x^3\ \ln x\ dx = \left[\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\,\frac{1}{x}\,dx$$

$$= \left[\frac{x^4}{4}\ln x\right]_1^e - \frac{1}{4}\int_1^e x^3\ dx$$

$$= \frac{1}{4}[e^4\ln e - \ln 1] - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^e$$

$$= \frac{e^4}{4} - \frac{1}{16}(e^4-1) = \frac{3e^4+1}{16}$$

b) .

$$\int x\sec^2 x\,dx\ \ ;\ \ \text{Let: } u=x, du=dx;\ \ dv=\sec^2 x\,dx, v=\tan x$$

$$\therefore \int x\sec^2 x\,dx = x\ \tan x - \int \tan x\ dx$$

$$= x\ \tan x + \ln|\cos x| + C$$

c) .

$$\int x^2\,e^x dx\ \ ;\ \ \text{Let: } u=x^2, du=2x\,dx;\ \ dv=e^x dx, v=e^x$$

$$\therefore \int x^2\,e^x dx = x^2e^x - 2\int x\,e^x\,dx$$

Also:

$$\int x\, e^x\, dx = x\, e^x - \int e^x\, dx = x\, e^x - e^x + C$$

$$\therefore \int x^2\, e^x dx = x^2 e^x - 2\int x\, e^x\, dx = x^2 e^x - 2x\, e^x + 2e^x + C$$

d) .

$$\int 2x\ \sin^{-1}(x^2)\, dx;\ ;\ \text{Let: } u = \sin^{-1}(x^2)\,, du = \frac{2xdx}{\sqrt{1-x^4}};\ \ dv = 2x\, dx, v = x^2$$

$$\therefore \int 2x\ \sin^{-1}(x^2)\, dx = x^2 \sin^{-1}(x^2) - \int x^2 . \frac{2xdx}{\sqrt{1-x^4}}$$

$$= x^2 \sin^{-1}(x^2) - \int \frac{2x^3\, dx}{\sqrt{1-x^4}} = x^2 \sin^{-1}(x^2) - \int 2x^3 (1-x^4)^{-1/2} dx$$

$$= x^2 \sin^{-1}(x^2) + \sqrt{1-x^4} + C$$

e) .

$$\int e^x \cos x\, dx\ \ ;\ \ \text{Let: } u = e^x, du = e^x\, dx;\ \ dv = \cos x\, dx, v = \sin x$$

$$\therefore \int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\ dx$$

$$\text{Again, } \int e^x \sin x\ \ dx; \text{Let: } u = e^x, du = e^x\, dx;\ \ dv = \sin x\, dx, v = -\cos x$$

$$\therefore \int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$

$$2\int e^x \cos x\, dx = e^x(\sin x + \cos x)$$

$$\int e^x \cos x\, dx = \frac{e^x}{2}(\sin x + \cos x)$$

2. Tabular Integration

The integrals of the form: $\int f(x).g(x)dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts.

Example 1: the same previous example:

$$\int x^2\, e^x dx$$

Solution:

$f(x) = x^2$ and $g(x) = e^x$, we list:

f(x) and its derivatives		*g(x) and its integrals*
x^2	(+)	e^x
$2x$	(-)	e^x
2	(+)	e^x
0		e^x

$$\therefore \int x^2\, e^x dx = x^2 e^x - 2x\, e^x + 2e^x + C$$

Example 2: Evaluate the following integral:

$$\int x^3 \sin x\, dx$$

Solution:

f(x) and its derivatives		*g(x) and its integrals*
x^3	(+)	$\sin x$
$3x^2$	(-)	$-\cos x$
$6x$	(+)	$-\sin x$
6	(-)	$\cos x$
0		$\sin x$

$$\therefore \int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Example 3: Evaluate the following integral:

$$\int (x^2 - 5x)\, e^x dx$$

Solution:

f(x) and its derivatives		g(x) and its integrals
x^2-5x	(+)	e^x
$2x-5$	(-)	e^x
2	(+)	e^x
0		e^x

$$\therefore \int (x^2-5x)\, e^x dx = (x^2-5x)e^x-(2x-5)e^x+2e^x+C$$

Example 4: Evaluate the following integral:

$$\int_0^{\pi/2} x^3 \cos 2x \, dx$$

Solution:

f(x) and its derivatives		g(x) and its integrals
x^3	(+)	$\cos 2x$
$3x^2$	(-)	$\frac{\sin 2x}{2}$
$6x$	(+)	$-\frac{\cos 2x}{4}$
6	(-)	$-\frac{\sin 2x}{8}$
0		$\frac{\cos 2x}{16}$

$$\therefore \int_0^{\pi/2} x^3 \cos 2x \, dx = \left[x^3\frac{\sin 2x}{2}-3x^2\frac{\cos 2x}{4}-6x\frac{\sin 2x}{8}-\frac{6\cos 2x}{16}\right]_0^{\pi/2}$$

$$=\left(\frac{\pi^3}{6}\frac{\sin\pi}{2}-\frac{3\pi^2}{16}\cos\pi-3\pi\frac{\sin\pi x}{8}-\frac{3\cos\pi}{8}\right)-\left(0-0-0-\frac{3}{8}\right)=\frac{-3\pi^2}{16}+\frac{3}{4}=\frac{3(4-\pi^2)}{16}$$

3. Reduction Formula

Example 5: Evaluate the following integral:

$$\int \cos^n x \, dx$$

Solution: We may think that $\cos^n x$ as $\cos^{n-1} x \, . \cos x$. Then,

$$\text{Let: } u = \cos^{n-1} x \quad \text{and} \quad dv = \cos x \; dx$$

$$du = (n-1)\cos^{n-2} x \, .(-\sin x) \, dx \quad \text{and} \quad v = \sin x$$

Hence,

$$\int \cos^n x \, dx = \cos^{n-1} x \, . \sin x - \int \sin x \, . \; (n-1)\cos^{n-2} x \, .(-\sin x) \, dx$$

$$= \cos^{n-1} x \, . \sin x + (n-1) \int \sin^2 x \, . \; \cos^{n-2} x \; dx$$

$$= \cos^{n-1} x \, . \sin x + (n-1) \int (1 - \cos^2 x) . \; \cos^{n-2} x \; dx$$

$$= \cos^{n-1} x \, . \sin x + (n-1) \int \cos^{n-2} x \;\; dx - (n-1) \int \cos^n x$$

If we add:

$$(n-1) \int \cos^n x$$

To both sides of this equation, we obtain:

$$n \int \cos^n x = \cos^{n-1} x \, . \sin x + (n-1) \int \cos^{n-2} x \;\; dx$$

Then, we divide both sides of equation by *n*

$$\int \cos^n x = \frac{\cos^{n-1} x \, . \sin x}{n} + \frac{(n-1)}{n} \int \cos^{n-2} x \;\; dx$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that:

$$\int \cos^3 x = \frac{\cos^2 x \,.\sin x}{3} + \frac{2}{3}\int \cos x \quad dx$$

$$= \frac{1}{3}\cos^2 x \,.\sin x + \frac{2}{3}\sin x + C$$

Homework #11

1. Evaluate the following integrals using integration by parts.

$$\int x^5 e^x \;\; dx$$

$$\int x^3 \ln x \;\; dx$$

$$\int \sin^{-1} y \;\; dy$$

$$\int e^{-2x} \sin 2x \;\; dx$$

$$\int \sqrt{x}\left(\sin^{-1}\sqrt{x}\right) \;\; dx$$

2. Use integration by parts to establish the reduction formula:

$$\int x^n \cos x \, dx = x^n \sin x - n\int x^{n-1} \sin x \, dx$$

$$\int (\ln x)^n \, dx = x(\ln x)^n - n\int (\ln x)^{n-1} \, dx$$

3. Show that:

$$\int_0^{\pi/2} \sin^n x \;\; dx = \int_0^{\pi/2} \cos^n x \;\; dx$$