10 METHODS OF INTEGRATION

10.1 INTRODUCTION. THE BASIC FORMULAS

If we start with the constants and the seven familiar functions x, e^x, $\ln x$, $\sin x$, $\cos x$, $\sin^{-1} x$, and $\tan^{-1} x$, and go on to build all possible finite combinations of these by applying the algebraic operations and the process of forming a function of a function, then we generate the class of *elementary functions.* Thus,

$$\ln \left[\frac{\tan^{-1}(x^2 + 35x^3)}{e^x + \sin \sqrt{x^3 + 1}} \right]$$

is an elementary function. These functions are often said to have *closed form*, because they can be written down in explicit formulas involving only a finite number of familiar functions.

It is clear that the problem of calculating the derivative of an elementary function can always be solved by a systematic application of the rules developed in the preceding chapters, and this derivative is always an elementary function. However, the inverse problem of integration—which in general is much more important—is very different and has no such clear-cut solution.

As we know, the problem of calculating the indefinite integral of a function $f(x)$,

$$\int f(x)\, dx = F(x), \tag{1}$$

is equivalent to finding a function $F(x)$ such that

$$\frac{d}{dx} F(x) = f(x). \tag{2}$$

It is true that we have succeeded in integrating a good many elementary functions by inverting differentiation formulas. But this doesn't carry us very far, because it amounts to little more than calculating the integral (1) by knowing the answer (2) in advance.

The fact of the matter is this: There does not exist any systematic procedure that can always be applied to any elementary function and leads step by step to a guaranteed answer in closed form. Indeed, there may not even *be* an answer. For example, the function $f(x) = e^{-x^2}$ looks simple enough, but its integral

$$\int e^{-x^2}\, dx \tag{3}$$

cannot be calculated within the class of elementary functions. This assertion is more than merely a report on the present inability of mathematicians to integrate (3); it is a statement of a deep theorem, to the effect that no elementary function exists whose derivative is e^{-x^2}.*

If all this sounds discouraging, it shouldn't be. There is much more that can be done in the way of integration than we have suggested so far, and it is very important for students to acquire a certain amount of technical skill in carrying out integrations whenever they *are* possible. The fact that integration must be considered as more of an art than a systematic process really makes it more interesting than differentiation. It is more like solving puzzles, because there is less certainty and more scope for individual ingenuity. Many students find this an agreeable change from the cut-and-dried routines that make some parts of mathematics rather dull.

Since integration is differentiation read backwards, our starting point must be a short table of standard types of integrals obtained by inverting differentiation formulas as we have done in the previous chapters. Much more extensive tables than the one given below are available in libraries, and with the aid of these tables most of the problems in this chapter can be solved by merely looking them up. However, students should realize that if they follow such a course they will defeat the intended purpose of developing their own skills. For this reason we make no use of integral tables beyond the short list of 15 formulas given below. Instead, we urge students to concentrate their efforts on gaining a clear understanding of the various methods of integration and learning how to apply them.

In addition to the method of substitution, which is already familiar to us, there are three principal methods of integration to be studied in this chapter: reduction to trigonometric integrals, decomposition into partial fractions, and integration by parts. These methods enable us to transform a given integral in many ways. The object of these transformations is always to break up the given integral into a sum of simpler parts that can be integrated at once by means of familiar formulas. Students should therefore be certain that they have thoroughly memorized all the following basic formulas. These formulas should be so well learned that when one of them is needed it pops into the mind almost involuntarily, like the name of a friend.

1 $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + c \qquad (n \neq -1).$

2 $\int \dfrac{du}{u} = \ln u + c.$

*Let there be no misunderstanding. The indefinite integral (3) *does* exist, because the function $F(x)$ defined by

$$F(x) = \int_0^x e^{-t^2}\,dt$$

is a perfectly respectable function with the property that

$$\frac{d}{dx}F(x) = e^{-x^2}.$$

[See equations (12) and (13) in Section 6.7.] The difficulty is that it can be proved that there is no way of expressing $F(x)$ as an elementary function. Some of the facts in this interesting part of calculus are described in Appendix A.9.

3 $\int e^u \, du = e^u + c.$

4 $\int \cos u \, du = \sin u + c.$

5 $\int \sin u \, du = -\cos u + c.$

6 $\int \sec^2 u \, du = \tan u + c.$

7 $\int \csc^2 u \, du = -\cot u + c.$

8 $\int \sec u \tan u \, du = \sec u + c.$

9 $\int \csc u \cot u \, du = -\csc u + c.$

10 $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + c.$

11 $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + c.$

12 $\int \tan u \, du = -\ln (\cos u) + c.$

13 $\int \cot u \, du = \ln (\sin u) + c.$

14 $\int \sec u \, du = \ln (\sec u + \tan u) + c.$

15 $\int \csc u \, du = -\ln (\csc u + \cot u) + c.$

The last four formulas are new, and complete our list of the integrals of the six trigonometric functions. Formulas 12 and 13 can be found by a straightforward process:

$$\int \tan u \, du = \int \frac{\sin u \, du}{\cos u} = -\int \frac{d(\cos u)}{\cos u} = -\ln (\cos u) + c$$

and

$$\int \cot u \, du = \int \frac{\cos u \, du}{\sin u} = \int \frac{d(\sin u)}{\sin u} = \ln (\sin u) + c.$$

Many people find that the easiest way to remember these two formulas is to think of the process by which they are obtained. Formula 14 can be found by an ingenious trick: If we multiply the integrand by $1 = (\sec u + \tan u)/(\sec u + \tan u)$, then we obtain

$$\int \sec u \, du = \int \frac{(\sec u + \tan u) \sec u \, du}{\sec u + \tan u} = \int \frac{(\sec^2 u + \sec u \tan u) \, du}{\sec u + \tan u}$$

$$= \int \frac{d(\sec u + \tan u)}{\sec u + \tan u} = \ln (\sec u + \tan u) + c.$$

A similar trick yields formula 15.

We repeat: These 15 formulas constitute the foundation on which we build throughout this chapter, and they must be at our fingertips. Take 20 or 30 minutes to memorize them. And then tomorrow, when they have been partially forgotten, memorize them again. And so on. The effort will be well rewarded.

10.2 THE METHOD OF SUBSTITUTION

In the method of substitution we introduce the auxiliary variable u as a new symbol for part of the integrand in the hope that its differential du will account for some other part and thereby reduce the complete integral to an easily recognizable form. Success in the use of this method depends on choosing a fruitful substitution, and this in turn depends on the ability to see at a glance that part of the integrand is the derivative of some other part.

We give several examples to help students review the procedure and make certain that they fully understand it.

Example 1 Find $\int xe^{-x^2}\,dx$.

Solution If we put $u = -x^2$, then $du = -2x\,dx$, $x\,dx = -\frac{1}{2}\,du$, and therefore

$$\int xe^{-x^2}\,dx = -\tfrac{1}{2}\int e^u\,du = -\tfrac{1}{2}e^u = -\tfrac{1}{2}e^{-x^2} + c.$$

It will be noticed that we insert the constant of integration only in the last step. Strictly speaking, this is incorrect; but we willingly commit this minor error in order to avoid cluttering up the previous steps with repeated c's. We also point out that this integral is easy to calculate even though the similar integral $\int e^{-x^2}\,dx$ is impossible. The reason for this is clearly the presence of the factor x, which is essentially (that is, up to a constant factor) the derivative of the exponent $-x^2$.

Example 2 Find

$$\int \frac{\cos x\,dx}{\sqrt{1 + \sin x}}.$$

Solution Here we notice that $\cos x\,dx$ is the differential of $\sin x$, and also of $1 + \sin x$. Thus, if we put $u = 1 + \sin x$, then $du = \cos x\,dx$ and

$$\int \frac{\cos x\,dx}{\sqrt{1 + \sin x}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du$$

$$= \frac{u^{1/2}}{\frac{1}{2}} = 2\sqrt{u} = 2\sqrt{1 + \sin x} + c.$$

Example 3 Find

$$\int \frac{dx}{x \ln x}.$$

Solution The fact that dx/x is the differential of $\ln x$ suggests the substitution $u = \ln x$, so $du = dx/x$ and

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln u = \ln (\ln x) + c.$$

Example 4 Find

$$\int \frac{dx}{\sqrt{9 - 4x^2}}.$$

Solution Since $4x^2 = (2x)^2$ we put $u = 2x$, so that $du = 2dx$, $dx = \frac{1}{2}\,du$, and

$$\int \frac{dx}{\sqrt{9 - 4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{9 - u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{3} = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c.$$

Example 5 Find

$$\int \frac{x\,dx}{\sqrt{9 - 4x^2}}.$$

Solution Here the fact that the x in the numerator is essentially the derivative of the expression $9 - 4x^2$ inside the radical suggests the substitution $u = 9 - 4x^2$. Then $du = -8x\,dx$, and

$$\int \frac{x\,dx}{\sqrt{9 - 4x^2}} = -\frac{1}{8}\int \frac{du}{\sqrt{u}} = -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8}\,\frac{u^{1/2}}{\frac{1}{2}} = -\frac{1}{4}\sqrt{u} = -\frac{1}{4}\sqrt{9 - 4x^2} + c.$$

In any particular integration problem the choice of the substitution is a matter of trial and error guided by experience. If our first substitution doesn't work, we should feel no hesitation about discarding it and trying another. Example 5 is similar in appearance to Example 4 and it might be thought that the same substitution will work again, but in fact—as we have seen—it requires an entirely different substitution.

We remind students of the summary of the method of substitution given at the end of Section 5.3. Also, we repeat the justification of the method given there because we now wish to extend this method to cover the case of definite integrals as well.

We start with a complicated integral of the form

$$\int f[g(x)]g'(x)\,dx. \tag{1}$$

If we put $u = g(x)$, then $du = g'(x)\,dx$ and the integral takes the new form

$$\int f(u)\,du.$$

If we can integrate this, so that

$$\int f(u)\,du = F(u) + c, \tag{2}$$

then since $u = g(x)$ we ought to be able to integrate (1) by writing

$$\int f[g(x)]g'(x)\,dx = F[g(x)] + c. \tag{3}$$

All that is needed to justify our procedure is to notice that (3) is a correct result, because

$$\frac{d}{dx} F[g(x)] = F'[g(x)]g'(x) = f[g(x)]g'(x)$$

by the chain rule.

The method of substitution applies to definite integrals as well as indefinite integrals. The crucial requirement is that the limits of integration must be suitably changed when the substitution is made. This can be expressed as follows:

$$\int_a^b f[g(x)]g'(x)\,dx = \int_c^d f(u)\,du,$$

where $c = g(a)$ and $d = g(b)$. The proof uses (2) and (3) and two applications of the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F[g(b)] - F[g(a)] \\ &= F(d) - F(c) = \int_c^d f(u)\,du.\end{aligned}$$

Thus, once the original integral is changed into a simpler integral in the variable u, the numerical evaluation can be carried out entirely in terms of u, provided the limits of integration are also correctly changed.

Example 6 Compute

$$\int_0^{\pi/3} \frac{\sin x\,dx}{\cos^2 x}.$$

Solution We put $u = \cos x$, so that $du = -\sin x\,dx$. Observe that $u = 1$ when $x = 0$ and $u = \frac{1}{2}$ when $x = \pi/3$. By changing both the variable of integration and the limits of integration we obtain

$$\int_0^{\pi/3} \frac{\sin x\,dx}{\cos^2 x} = \int_1^{1/2} \frac{-du}{u^2} = \frac{1}{u}\bigg]_1^{1/2} = 2 - 1 = 1.$$

This technique removes the necessity of returning to the original variable in order to make the final numerical evaluation.

PROBLEMS

Find the following integrals.

1 $\int \sqrt{3 - 2x}\,dx.$

2 $\int \frac{2x\,dx}{(4x^2 - 1)^2}.$

3 $\int \frac{\ln x\,dx}{x[1 + (\ln x)^2]}.$

4 $\int \cos x\, e^{\sin x}\,dx.$

5 $\int \sin 2x\,dx.$

6 $\int \frac{x\,dx}{\sqrt{16 - x^4}}.$

7 $\int \cot(3x - 1)\,dx.$

8 $\int \sin x \cos x\,dx.$

9 $\int x\sqrt{x^2 + 1}\,dx.$

10 $\int \frac{dx}{x + 2}.$

11 $\int e^{5x}\,dx.$

12 $\int x \cos x^2\,dx.$

13 $\int \csc^2(3x + 2)\,dx.$

14 $\int \frac{dx}{x^2 + 16}.$

15 $\int_{-3}^{1} \frac{dx}{\sqrt{3 - 2x}}.$

16 $\int (x^3 + 1)^2\,dx.$

17 $\int \frac{\sin x\,dx}{\sqrt{1 - \cos x}}.$

18 $\int \frac{(2x + 1)\,dx}{x^2 + x + 2}.$

19 $\int \frac{e^{\tan^{-1} x}}{1 + x^2}\,dx.$

20 $\int \frac{\sin \sqrt{x}}{\sqrt{x}}\,dx.$

21 $\int \sec 5x \tan 5x \, dx$.

22 $\int \dfrac{dx}{x\sqrt{\ln x}}$.

23 $\int \dfrac{\ln x \, dx}{x}$.

24 $\int \dfrac{\sin x \, dx}{\cos^2 x}$.

25 $\int_0^{\pi/2} \dfrac{\cos x \, dx}{1 + \sin x}$.

26 $\int \cos 3x \, dx$.

27 $\int \dfrac{e^x \, dx}{\sqrt{1 - e^{2x}}}$.

28 $\int \dfrac{dx}{\cos 2x}$.

29 $\int \sin^2 x \cos x \, dx$.

30 $\int_0^3 \tan^2 \frac{1}{3}x \sec^2 \frac{1}{3}x \, dx$.

31 $\int \dfrac{e^x \, dx}{1 + e^x}$.

32 $\int \dfrac{\cos (\ln x) \, dx}{x}$.

33 $\int \tan 3x \, dx$.

34 $\int \dfrac{\sec^2 x \, dx}{\sqrt{1 + \tan x}}$.

35 $\int \dfrac{4x \, dx}{\sqrt{x^2 + 1}}$.

36 $\int \dfrac{e^{\sqrt{x}} \, dx}{\sqrt{x}}$.

37 $\int \dfrac{e^x \, dx}{1 + e^{2x}}$.

38 $\int \dfrac{\sin^{-1} x \, dx}{\sqrt{1 - x^2}}$.

39 $\int (e^x + 1)^6 e^x \, dx$.

40 $\int 6x^2 e^{-x^3} \, dx$.

41 $\int \sec^2 5x \, dx$.

42 $\int \cot 4x \, dx$.

43 $\int \csc 2x \cot 2x \, dx$.

44 $\int_2^3 \dfrac{2x \, dx}{x^2 - 3}$.

Compute each of the following definite integrals by making a suitable substitution and changing the limits of integration.

45 $\int_1^2 \dfrac{(2x + 1) \, dx}{\sqrt{x^2 + x + 2}}$.

46 $\int_0^{\pi/4} \tan^2 x \sec^2 x \, dx$.

47 $\int_1^e \dfrac{\sqrt{\ln x} \, dx}{x}$.

48 $\int_0^{\pi/3} \sec^3 x \tan x \, dx$.

49 Each of the following integrals is easy to compute for a particular value of n. Find this value and carry out the integration. For example, $\int x^n \sin x^2 \, dx$ is easily computed for $n = 1$:

$$\int x \sin x^2 \, dx = -\tfrac{1}{2} \cos x^2 + c.$$

(a) $\int x^n e^{x^4} \, dx$. (b) $\int x^n \cos x^3 \, dx$.
(c) $\int x^n \ln x \, dx$. (d) $\int x^n \sec^2 \sqrt{x} \, dx$.

50 The derivation given in the text for formula 14 is somewhat tainted by rabbit-out-of-the-hat trickery. Derive this formula in a more reasonable way by using

$$\int \sec u \, du = \int \frac{du}{\cos u} = \int \frac{\cos u \, du}{\cos^2 u} = \int \frac{\cos u \, du}{1 - \sin^2 u}$$

to write the given integral as an integral of the form $\int du/(1 - u^2)$, and then use

$$\frac{1}{1 - u^2} = \frac{1}{2}\left(\frac{1}{1 + u} + \frac{1}{1 - u}\right).$$

51 Give a similar derivation for formula 15.

10.3 CERTAIN TRIGONOMETRIC INTEGRALS

In the next two sections we discuss several methods for reducing a given integral to one involving trigonometric functions. It will therefore be useful to increase our ability to calculate such trigonometric integrals.

A power of a trigonometric function multiplied by its differential is easy to integrate. Thus,

$$\int \sin^3 x \cos x \, dx = \int \sin^3 x \, d(\sin x) = \tfrac{1}{4} \sin^4 x + c$$

and

$$\int \tan^2 x \sec^2 x \, dx = \int \tan^2 x \, d(\tan x) = \tfrac{1}{3} \tan^3 x + c.$$

Other trigonometric integrals can often be reduced to problems of this type by using appropriate trigonometric identities.

We begin by considering integrals of the form

$$\int \sin^m x \cos^n x \, dx, \tag{1}$$

where one of the exponents is an odd positive integer. If n is odd, we factor out $\cos x\, dx$, which is $d(\sin x)$; and since an even power of $\cos x$ remains, we can use the identity $\cos^2 x = 1 - \sin^2 x$ to express the remaining part of the integrand entirely in terms of $\sin x$. And if m is odd, we factor out $\sin x\, dx$, which is $-d(\cos x)$, and use the identity $\sin^2 x = 1 - \cos^2 x$ in a similar way. The following two examples illustrate the procedure.

Example 1

$$\begin{aligned}\int \sin^2 x \cos^3 x\, dx &= \int \sin^2 x \cos^2 x \cos x\, dx \\ &= \int \sin^2 x(1 - \sin^2 x)\, d(\sin x) \\ &= \int (\sin^2 x - \sin^4 x)\, d(\sin x) \\ &= \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + c.\end{aligned}$$

Example 2

$$\begin{aligned}\int \sin^3 x\, dx &= \int \sin^2 x \sin x\, dx \\ &= -\int (1 - \cos^2 x)\, d(\cos x) \\ &= -\cos x + \tfrac{1}{3}\cos^3 x + c.\end{aligned}$$

If one of the exponents in (1) is an odd positive integer that is quite large, it may be necessary to use the binomial theorem, and in such a case an explicit use of the method of substitution may be desirable for the sake of clarity. For instance, every odd positive power of $\cos x$, whether large or small, has the form

$$\cos^{2n+1} x = \cos^{2n} x \cos x = (\cos^2 x)^n \cos x = (1 - \sin^2 x)^n \cos x,$$

where n is a nonnegative integer. If we put $u = \sin x$ and $du = \cos x\, dx$, then

$$\begin{aligned}\int \cos^{2n+1} x\, dx &= \int (1 - \sin^2 x)^n \cos x\, dx \\ &= \int (1 - u^2)^n\, du.\end{aligned}$$

If necessary, the expression $(1 - u^2)^n$ can now be expanded by applying the binomial theorem, and the resulting polynomial in u is easy to integrate term by term.

If both exponents in (1) are nonnegative even integers, then it is necessary to change the form of the integrand by using the half-angle formulas

$$\cos^2 \theta = \tfrac{1}{2}(1 + \cos 2\theta) \qquad \text{and} \qquad \sin^2 \theta = \tfrac{1}{2}(1 - \cos 2\theta). \tag{2}$$

We hope students have thoroughly memorized these important formulas, but if they are forgotten they can easily be recovered by adding and subtracting the identities

$$\cos^2\theta + \sin^2\theta = 1,$$
$$\cos^2\theta - \sin^2\theta = \cos 2\theta.$$

The uses of (2) are shown in the following examples.

Example 3 The half-angle formula for the cosine enables us to write

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1 + \cos 2x)\,dx = \tfrac{1}{2}\int dx + \tfrac{1}{2}\int \cos 2x\,dx$$
$$= \tfrac{1}{2}x + \tfrac{1}{4}\int \cos 2x\,d(2x) = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c.$$

If we wish to express this result in terms of the variable x (instead of $2x$), we use the double-angle formula $\sin 2x = 2 \sin x \cos x$ and write

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{2}\sin x \cos x + c.$$

Example 4 Two successive applications of the half-angle formula for the cosine give

$$\cos^4 x = (\cos^2 x)^2 = \tfrac{1}{4}(1 + \cos 2x)^2 = \tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$
$$= \tfrac{1}{4}[1 + 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)]$$
$$= \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x,$$

so

$$\int \cos^4 x\,dx = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + c.$$

As these examples show, the value of the half-angle formulas (2) for this work lies in the fact that they allow us to reduce the exponent by a factor of $\frac{1}{2}$ at the expense of multiplying the angle by 2, which is a considerable advantage purchased at very low cost.

Example 5 By using both of the half-angle formulas we get

$$\int \sin^2 x \cos^2 x\,dx = \int \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2}\,dx$$
$$= \tfrac{1}{4}\int (1 - \cos^2 2x)\,dx = \tfrac{1}{4}\int [1 - \tfrac{1}{2}(1 + \cos 4x)]\,dx$$
$$= \tfrac{1}{8}\int dx - \tfrac{1}{8}\int \cos 4x\,dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + c.$$

We can also find this integral by combining the results of Examples 3 and 4:

$$\int \sin^2 x \cos^2 x\,dx = \int (1 - \cos^2 x)\cos^2 x\,dx$$
$$= \int \cos^2 x\,dx - \int \cos^4 x\,dx$$
$$= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x - \tfrac{1}{32}\sin 4x$$
$$= \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + c.$$

We next consider integrals of the form

$$\int \tan^m x \sec^n x \, dx,$$

where n is an even positive integer or m is an odd positive integer. Our work is based on the fact that $d(\tan x) = \sec^2 x \, dx$ and $d(\sec x) = \sec x \tan x \, dx$, and we exploit the identity $\tan^2 x + 1 = \sec^2 x$. An example illustrating each case will be enough to show the general method.

Example 6

$$\begin{aligned}\int \tan^4 x \sec^6 x \, dx &= \int \tan^4 x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^4 x \, (\tan^2 x + 1)^2 \, d(\tan x) \\ &= \int \tan^4 x \, (\tan^4 x + 2 \tan^2 x + 1) \, d(\tan x) \\ &= \int (\tan^8 x + 2 \tan^6 x + \tan^4 x) \, d(\tan x) \\ &= \tfrac{1}{9} \tan^9 x + \tfrac{2}{7} \tan^7 x + \tfrac{1}{5} \tan^5 x + c.\end{aligned}$$

Example 7

$$\begin{aligned}\int \tan^3 x \sec^5 x \, dx &= \int \tan^2 x \sec^4 x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1) \sec^4 x \, d(\sec x) \\ &= \int (\sec^6 x - \sec^4 x) \, d(\sec x) \\ &= \tfrac{1}{7} \sec^7 x - \tfrac{1}{5} \sec^5 x + c.\end{aligned}$$

In essentially the same way we can handle integrals of the form

$$\int \cot^m x \csc^n x \, dx,$$

where n is an even positive integer or m is an odd positive integer. Our tools in these cases are the formulas $d(\cot x) = -\csc^2 x \, dx$ and $d(\csc x) = -\csc x \cot x \cdot dx$, and when necessary we use the identity $1 + \cot^2 x = \csc^2 x$.

Another approach to trigonometric integrals that is sometimes useful is to express each function occurring in the integral in terms of sines and cosines alone.

Example 8 We already know from our work with derivatives that

$$\int \sec x \tan x \, dx = \sec x + c.$$

However, this formula can also be obtained directly, by writing

$$\int \sec x \tan x \, dx = \int \frac{1}{\cos x} \frac{\sin x}{\cos x} \, dx = \int \frac{\sin x \, dx}{\cos^2 x}.$$

If we now put $u = \cos x$ and $du = -\sin x\, dx$, then we get

$$\int \sec x \tan x\, dx = \int \frac{\sin x\, dx}{\cos^2 x}$$

$$= \int \frac{-du}{u^2} = \frac{1}{u} = \frac{1}{\cos x} = \sec x + c.$$

PROBLEMS

Find each of the following integrals.

1 $\int \sin^2 x\, dx.$

2 $\int \sin^4 x\, dx.$

3 $\int \cos^6 x\, dx.$

4 $\int \cos^2 3x\, dx.$

5 $\int \sin^3 x \cos^2 x\, dx.$

6 $\int \sin^2 x \cos^5 x\, dx.$

7 $\int \cos^3 x\, dx.$

8 $\int_0^{\pi/2} \sin^3 x \cos^3 x\, dx.$

9 $\int \sqrt{\sin x} \cos^3 x\, dx.$

10 $\int \sin^3 5x \cos 5x\, dx.$

11 $\int \sin^2 3x \cos^2 3x\, dx.$

12 $\int \frac{dx}{\sin x \cos x}.$

13 $\int_0^{\pi/4} \sec^4 x\, dx.$

14 $\int \frac{dx}{\cos^2 x}.$

15 $\int \tan^5 x \sec^3 x\, dx.$

16 $\int \csc^4 x\, dx.$

17 $\int \cot^2 x\, dx.$

18 $\int \cot^3 x\, dx.$

19 $\int \frac{dx}{\sin^2 4x}.$

20 $\int \cot^2 5x \csc^4 5x\, dx.$

21 $\int \frac{1 + \cos 2x}{\sin^2 2x}\, dx.$

22 $\int \tan^2 x \cos x\, dx.$

23 $\int \sin 3x \cot 3x\, dx.$

24 Find $\int \tan x\, dx$ (which we already know) by the method of Example 7.

25 Use the identity $\tan^2 x = \sec^2 x - 1$ to find
(a) $\int \tan^2 x\, dx$, $\int \tan^4 x\, dx$, $\int \tan^6 x\, dx$;
(b) $\int \tan^3 x\, dx$, $\int \tan^5 x\, dx$, $\int \tan^7 x\, dx$.

26 If n is any positive integer ≥ 2, show that

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$$

This is called a *reduction formula*, because it reduces the problem of integrating $\tan^n x$ to the problem of integrating $\tan^{n-2} x$.

27 Find the volume of the solid of revolution generated when the indicated region under each of the following curves is revolved about the x-axis:
(a) $y = \sin x$, $0 \leq x \leq \pi$;
(b) $y = \sec x$, $0 \leq x \leq \pi/4$;
(c) $y = \tan 2x$, $0 \leq x \leq \pi/8$;
(d) $y = \cos^2 x$, $\pi/2 \leq x \leq \pi$.

28 Find the length of the curve $y = \ln(\cos x)$ between $x = 0$ and $x = \pi/4$.

29 Find $\int \sec^3 x\, dx$ by exploiting the observation that $\sec^3 x$ will clearly appear in the derivative of $\sec x \tan x$.

30 Find $\int \csc^3 x\, dx$ by adapting the idea suggested for Problem 29.

10.4 TRIGONOMETRIC SUBSTITUTIONS

An integral involving one of the radical expressions $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ (where a is a positive constant) can often be transformed into a familiar trigonometric integral by using a suitable trigonometric substitution or change of variable.

There are three cases, which depend on the trigonometric identities

$$1 - \sin^2 \theta = \cos^2 \theta, \tag{1}$$

$$1 + \tan^2 \theta = \sec^2 \theta, \tag{2}$$

$$\sec^2 \theta - 1 = \tan^2 \theta. \tag{3}$$

If the given integral involves $\sqrt{a^2 - x^2}$, then changing the variable from x to θ by writing

$$x = a \sin \theta \quad \text{replaces} \quad \sqrt{a^2 - x^2} \quad \text{by} \quad a \cos \theta, \tag{4}$$

because $a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta$. Similarly, if the given integral involves $\sqrt{a^2 + x^2}$, then by identity (2) we see that the substitution

$$x = a \tan \theta \qquad \text{replaces} \qquad \sqrt{a^2 + x^2} \qquad \text{by} \qquad a \sec \theta, \tag{5}$$

because $a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta$; and if it involves $\sqrt{x^2 - a^2}$, then by identity (3) the substitution

$$x = a \sec \theta \qquad \text{replaces} \qquad \sqrt{x^2 - a^2} \qquad \text{by} \qquad a \tan \theta, \tag{6}$$

because $x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta$. We illustrate these procedures as follows.

Example 1 Find

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx.$$

Solution This integral is of the first type, so we write

$$x = a \sin \theta, \qquad dx = a \cos \theta \, d\theta, \qquad \sqrt{a^2 - x^2} = a \cos \theta.$$

Then

$$\begin{aligned} \int \frac{\sqrt{a^2 - x^2}}{x}\, dx &= \int \frac{a \cos \theta}{a \sin \theta} a \cos \theta \, d\theta = a \int \frac{\cos^2 \theta}{\sin \theta}\, d\theta \\ &= a \int \frac{1 - \sin^2 \theta}{\sin \theta}\, d\theta = a \int (\csc \theta - \sin \theta)\, d\theta \\ &= -a \ln (\csc \theta + \cot \theta) + a \cos \theta. \end{aligned} \tag{7}$$

This completes the integration, and we now must write the answer in terms of the original variable x. We do this quickly and easily by drawing a right triangle (Fig. 10.1) whose sides are labeled in the simplest way that is consistent with the equation $x = a \sin \theta$ or $\sin \theta = x/a$. This figure tells us at once that

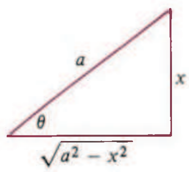

Figure 10.1

$$\csc \theta = \frac{a}{x}, \qquad \cot \theta = \frac{\sqrt{a^2 - x^2}}{x}, \qquad \text{and} \qquad \cos \theta = \frac{\sqrt{a^2 - x^2}}{a},$$

so from (7) we have

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = \sqrt{a^2 - x^2} - a \ln \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right) + c.$$

Example 2 Find

$$\int \frac{dx}{\sqrt{a^2 + x^2}}.$$

Solution Here we have an integral of the second type, so we write

$$x = a \tan \theta, \qquad dx = a \sec^2 \theta \, d\theta, \qquad \sqrt{a^2 + x^2} = a \sec \theta.$$

This yields

$$\begin{aligned} \int \frac{dx}{\sqrt{a^2 + x^2}} &= \int \frac{a \sec^2 \theta \, d\theta}{a \sec \theta} = \int \sec \theta \, d\theta \\ &= \ln (\sec \theta + \tan \theta). \end{aligned} \tag{8}$$

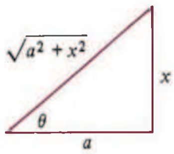

Figure 10.2

The substitution equation $x = a \tan\theta$ or $\tan\theta = x/a$ is pictured in Fig. 10.2, and from this figure we obtain

$$\sec\theta = \frac{\sqrt{a^2+x^2}}{a} \quad \text{and} \quad \tan\theta = \frac{x}{a}.$$

We therefore continue the calculation in (8) by writing

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(\frac{\sqrt{a^2+x^2}+x}{a}\right) + c' \tag{9}$$

$$= \ln(\sqrt{a^2+x^2}+x) + c. \tag{10}$$

Students will notice that since

$$\ln\left(\frac{\sqrt{a^2+x^2}+x}{a}\right) = \ln(\sqrt{a^2+x^2}+x) - \ln a,$$

the constant $-\ln a$ has been grouped together with the constant of integration c', and the quantity $-\ln a + c'$ is then rewritten as c. Usually we don't bother to make notational distinctions between one constant of integration and another, because all are completely arbitrary; but we do so here in the hope of clarifying the transition from (9) to (10).

Example 3 Find

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx.$$

Solution This integral is of the third type, so we write

$$x = a\sec\theta, \qquad dx = a\sec\theta\tan\theta\,d\theta, \qquad \sqrt{x^2-a^2} = a\tan\theta.$$

Then

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \int \frac{a\tan\theta}{a\sec\theta}\,a\sec\theta\tan\theta\,d\theta$$

$$= a\int \tan^2\theta\,d\theta = a\int(\sec^2\theta - 1)\,d\theta$$

$$= a\tan\theta - a\theta.$$

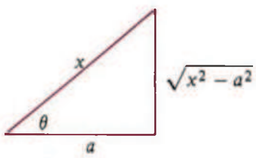

Figure 10.3

In this case our substitution equation $\sec\theta = x/a$ is portrayed in Fig. 10.3, which tells us that

$$\tan\theta = \frac{\sqrt{x^2-a^2}}{a} \quad \text{and} \quad \theta = \tan^{-1}\frac{\sqrt{x^2-a^2}}{a}.$$

The desired integral can therefore be written as

$$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\tan^{-1}\frac{\sqrt{x^2-a^2}}{a} + c.$$

There is one feature of these calculations that we have not taken into account. In (4) we tacitly wrote

$$\sqrt{1 - \sin^2 \theta} = \cos \theta$$

without checking the correctness of the algebraic sign. This was careless, because $\cos \theta$ is sometimes negative and sometimes positive. However, the variable θ, which in this case is $\sin^{-1} x/a$, is restricted to the interval $-\pi/2 \leq \theta \leq \pi/2$, and on this interval $\cos \theta$ is nonnegative, as we assumed. Similar comments apply to the substitutions (5) and (6).

Example 4 As a concrete illustration of the use of these methods, we determine the equation of the *tractrix*. This famous curve can be defined as follows: It is the path of an object dragged along a horizontal plane by a string of constant length when the other end of the string moves along a straight line in the plane. (The word "tractrix" comes from the Latin *tractere,* meaning "to drag.")

Suppose the plane is the xy-plane and the object starts at the point $(a, 0)$ with the other end of the string at the origin. If this end moves up the y-axis as shown on the left in Fig. 10.4, then the string is always tangent to the curve, and the length of the tangent between the y-axis and the point of contact is always equal to a. The slope of the tangent is therefore given by the formula

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x};$$

and by separating the variables and using the result of Example 1, we have

$$y = -\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = a \ln\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2} + c.$$

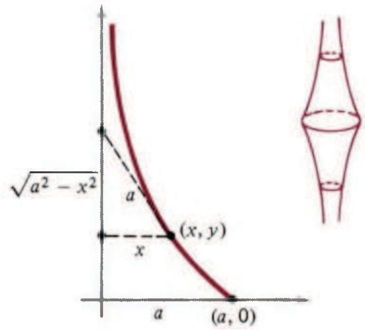

Figure 10.4

Since $y = 0$ when $x = a$, we see that $c = 0$, so

$$y = a \ln\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}$$

is the equation of the tractrix, or at least of the part shown in the figure.

If the end of the string moves down the y-axis, then another part of the curve is generated; and if these two parts are revolved about the y-axis, the resulting "double-trumpet" surface shown on the right in Fig. 10.4 is called a *pseudosphere.* In the branch of mathematics concerned with the geometry of curved surfaces, the pseudosphere is a model for Lobachevsky's version of non-Euclidean geometry. It is a surface of constant negative curvature, and the sum of the angles of any triangle on the surface is less than 180°.

Another famous curve whose equation can be determined by these methods of integration is the *catenary,* which is the curve assumed by a flexible chain or cable hanging between two fixed points. The details are a bit complicated, so we give a derivation in Appendix 1 at the end of this chapter for students who have chosen to omit the optional Section 9.7.

The substitution procedures described in this section can be given a general justification or proof similar to that provided in Section 10.2. Students who are interested in such matters will find the details in Appendix A.10.

PROBLEMS

Find each of the following integrals.

1 $\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx.$

2 $\int \frac{x^2\,dx}{\sqrt{4 - x^2}}.$

3 $\int \frac{dx}{(a^2 + x^2)^2}.$

4 $\int \frac{dx}{x^2\sqrt{a^2 + x^2}}.$

5 $\int \frac{x^3\,dx}{\sqrt{9 - x^2}}.$

6 $\int \frac{dx}{x\sqrt{a^2 - x^2}}.$

7 $\int \frac{dx}{x\sqrt{a^2 + x^2}}.$

8 $\int \frac{dx}{x + x^3}.$

9 $\int \frac{dx}{\sqrt{x^2 - a^2}}.$

10 $\int \frac{dx}{x^3\sqrt{x^2 - a^2}}.$

11 $\int \sqrt{a^2 + x^2}\,dx.$*

12 $\int \frac{x^3\,dx}{a^2 + x^2}.$

13 $\int \frac{dx}{a^2 - x^2}.$

14 $\int \frac{dx}{(a^2 - x^2)^{3/2}}.$

15 $\int \frac{\sqrt{a^2 + x^2}}{x}\,dx.$

16 $\int x^3\sqrt{a^2 + x^2}\,dx.$

17 $\int \frac{\sqrt{x^2 - a^2}}{x^2}\,dx.$

18 $\int \frac{dx}{(x^2 - a^2)^{3/2}}.$

19 $\int x^2\sqrt{a^2 - x^2}\,dx.$

20 $\int (1 - 4x^2)^{3/2}\,dx.$

The following integrals would normally be found in a different way, but this time work them out by using trigonometric substitutions.

21 $\int \frac{x\,dx}{\sqrt{4 - x^2}}.$

22 $\int \frac{x\,dx}{(a^2 - x^2)^{3/2}}.$

23 $\int \frac{dx}{a^2 + x^2}.$

24 $\int \frac{x\,dx}{4 + x^2}.$

*Hint: See Problem 29 in Section 10.3.

25 $\int x\sqrt{9 - x^2}\,dx.$

26 $\int \frac{dx}{\sqrt{a^2 - x^2}}.$

27 $\int \frac{x\,dx}{\sqrt{9 + x^2}}.$

28 $\int \frac{x\,dx}{\sqrt{x^2 - 4}}.$

29 Use integration to show that the area of a circle of radius a is πa^2.

30 In a circle of radius a, a chord b units from the center cuts off a chunk of the circle called a *segment*. Find a formula for the area of this segment.

31 If the circle $(x - b)^2 + y^2 = a^2$ $(0 < a < b)$ is revolved about the y-axis, the resulting solid of revolution is called a *torus* (see Problem 11 in Section 7.3). Use the shell method to find the volume of this torus.

32 Find the length of the parabola $y = x^2$ between $x = 0$ and $x = 1$. Hint: Use the result of Problem 29 in Section 10.3.

33 Find the length of the curve $y = \ln x$ between $x = 1$ and $x = \sqrt{8}$.

34 The given region under each of the following curves is revolved about the x-axis. Find the volume of the solid of revolution.

(a) $y = \frac{x^{3/2}}{\sqrt{x^2 + 4}}$ between $x = 0$ and $x = 4$.

(b) $y = \frac{1}{x^2 + 1}$ between $x = 0$ and $x = 1$.

(c) $y = \sqrt{4 - x^2}$ between $x = 1$ and $x = 2$.

35 The curve $\frac{1}{2}x^2 + y^2 = 1$ is an ellipse. Sketch the graph and show that its complete length equals the length of one cycle of $y = \sin x$. (This integral is a so-called *elliptic integral*, and is known to be impossible to evaluate in terms of elementary functions. For more details see Appendix A.9.)

10.5 COMPLETING THE SQUARE

In Section 10.4 we used trigonometric substitutions to calculate integrals containing $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$. By the algebraic device of completing the square, we can extend these methods to integrals involving general quadratic polynomials and their square roots, that is, expressions of the form $ax^2 + bx + c$ and $\sqrt{ax^2 + bx + c}$. We remind students that the process of completing the square is based on the simple fact that

$$(x + A)^2 = x^2 + 2Ax + A^2;$$

this tells us that the right side is a perfect square (the square of $x + A$) because its constant term is in the square of half the coefficient of x.

Example 1 Find

$$\int \frac{(x+2)\,dx}{\sqrt{3+2x-x^2}}.$$

Solution Since the coefficient of the term x^2 under the radical is negative, we place the terms containing x in parentheses preceded by a minus sign, leaving space for completing the square,

$$\begin{aligned} 3+2x-x^2 = 3-(x^2-2x+\quad) &= 4-(x^2-2x+1) \\ &= 4-(x-1)^2 = a^2-u^2, \end{aligned}$$

where $u = x-1$ and $a = 2$. Since $x = u+1$, we have $dx = du$ and $x+2 = u+3$, and therefore

$$\begin{aligned} \int \frac{(x+2)\,dx}{\sqrt{3+2x-x^2}} &= \int \frac{(u+3)\,du}{\sqrt{a^2-u^2}} = \int \frac{u\,du}{\sqrt{a^2-u^2}} + 3\int \frac{du}{\sqrt{a^2-u^2}} \\ &= -\sqrt{a^2-u^2} + 3\sin^{-1}\frac{u}{a} \\ &= -\sqrt{3+2x-x^2} + 3\sin^{-1}\left(\frac{x-1}{2}\right) + c. \end{aligned}$$

Example 2 Find

$$\int \frac{dx}{x^2+2x+10}.$$

Solution We complete the square on the terms containing x, and write

$$\begin{aligned} x^2+2x+10 = (x^2+2x+\quad)+10 &= (x^2+2x+1)+9 \\ &= (x+1)^2+9 = u^2+a^2, \end{aligned}$$

where $u = x+1$ and $a = 3$. We now have $du = dx$ or $dx = du$, so

$$\begin{aligned} \int \frac{dx}{x^2+2x+10} &= \int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} \\ &= \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + c. \end{aligned}$$

Example 3 Find

$$\int \frac{x\,dx}{\sqrt{x^2-2x+5}}.$$

Solution We write

$$\begin{aligned} x^2-2x+5 = (x^2-2x+\quad)+5 &= (x^2-2x+1)+4 \\ &= (x-1)^2+4 = u^2+a^2, \end{aligned}$$

where $u = x-1$ and $a = 2$. Then $x = u+1$, $dx = du$, and we have

$$\int \frac{x\,dx}{\sqrt{x^2-2x+5}} = \int \frac{(u+1)\,du}{\sqrt{u^2+a^2}} = \int \frac{u\,du}{\sqrt{u^2+a^2}} + \int \frac{du}{\sqrt{u^2+a^2}}.$$

The second integral here is the one considered in Example 2 in Section 10.4, so we have

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln (u + \sqrt{u^2 + a^2}),$$

and therefore

$$\int \frac{x\,dx}{\sqrt{x^2 - 2x + 5}} = \sqrt{u^2 + a^2} + \ln (u + \sqrt{u^2 + a^2})$$
$$= \sqrt{x^2 - 2x + 5} + \ln (x - 1 + \sqrt{x^2 - 2x + 5}) + c.$$

Example 4 Find

$$\int \frac{dx}{\sqrt{x^2 - 4x - 5}}.$$

Solution Here we have

$$x^2 - 4x - 5 = (x^2 - 4x + \quad) - 5 = (x^2 - 4x + 4) - 9$$
$$= (x - 2)^2 - 9 = u^2 - a^2,$$

where $u = x - 2$ and $a = 3$. By using the result of Problem 9 in Section 10.4 (or by quickly working out the necessary formula again by putting $u = a \sec \theta$) we complete the calculation as follows:

$$\int \frac{dx}{\sqrt{x^2 - 4x - 5}} = \int \frac{du}{\sqrt{u^2 - a^2}} = \ln (u + \sqrt{u^2 - a^2})$$
$$= \ln (x - 2 + \sqrt{x^2 - 4x - 5}) + c.$$

If an integral involves the square root of a third-, fourth-, or higher-degree polynomial, then it can be proved that there does not exist any general method for carrying out the integration. A few integrals of this kind are discussed in Appendix A.9.

PROBLEMS

Calculate the following integrals.

1 $\int \frac{dx}{\sqrt{2x - x^2}}$.

2 $\int \frac{dx}{\sqrt{5 + 4x - x^2}}$.

3 $\int \frac{dx}{x^2 + 4x + 5}$.

4 $\int \frac{dx}{x^2 - x + 1}$.

5 $\int \frac{(x + 1)\,dx}{\sqrt{2x - x^2}}$.

6 $\int \frac{(x + 3)\,dx}{\sqrt{5 + 4x - x^2}}$.

7 $\int \frac{x^2\,dx}{\sqrt{6x - x^2}}$.

8 $\int \frac{(x - 1)\,dx}{\sqrt{x^2 + 4x + 5}}$.

9 $\int \frac{(x + 7)\,dx}{x^2 + 2x + 5}$.

10 $\int \frac{\sqrt{x^2 + 2x - 3}}{x + 1}\,dx$.

11 $\int \frac{dx}{\sqrt{x^2 - 2x - 8}}$.

12 $\int \frac{dx}{\sqrt{5 + 3x - 2x^2}}$.

13 $\int \frac{dx}{\sqrt{4x^2 + 4x + 17}}$.

14 $\int \frac{(4x + 3)\,dx}{(x^2 - 2x + 2)^{3/2}}$.

15 $\int \frac{dx}{(x^2 - 2x - 3)^{3/2}}$.

16 $\int \frac{dx}{(x + 2)\sqrt{x^2 + 4x + 3}}$.

10.6 THE METHOD OF PARTIAL FRACTIONS

We recall that a rational function is a quotient of two polynomials. By taking the denominator of such a quotient to be 1, we see that the polynomials themselves are included among the rational functions. As we know, the simple rational functions

$$2x+1, \quad \frac{1}{x^2}, \quad \frac{1}{x}, \quad \frac{x}{x^2+1}, \quad \text{and} \quad \frac{1}{x^2+1}$$

have the following integrals:

$$x^2+x, \quad -\frac{1}{x}, \quad \ln x, \quad \tfrac{1}{2}\ln(x^2+1), \quad \text{and} \quad \tan^{-1} x.$$

Our purpose in this section is to describe a systematic procedure for computing the integral of any rational function, and we shall find that this integral can always be expressed in terms of polynomials, rational functions, logarithms, and inverse tangents. The basic idea is to break up a given rational function into a sum of simpler fractions (called *partial fractions*) which can be integrated by methods discussed earlier.

A rational function is called *proper* if the degree of the numerator is less than the degree of the denominator. Otherwise, it is said to be *improper.* For example,

$$\frac{x}{(x-1)(x+2)^2} \quad \text{and} \quad \frac{x^2+2}{x(x^2-9)}$$

are proper, while

$$\frac{x^4}{x^4-1} \quad \text{and} \quad \frac{2x^3-3x^2+2x-4}{x^2+4}$$

are improper. If we have to integrate an improper rational function, it is essential to begin by performing long division until we reach a remainder whose degree is less than that of the denominator. We illustrate with the second improper rational function just mentioned. Long division yields

$$\begin{array}{r|l}
 & 2x - 3 \\ \hline
x^2+4 & 2x^3 - 3x^2 + 2x - 4 \\
 & \underline{2x^3 \qquad\quad + 8x} \\
 & \quad -3x^2 - 6x - 4 \\
 & \quad \underline{-3x^2 \qquad - 12} \\
 & \qquad\quad -6x + 8
\end{array}$$

This means that the rational function in question can be written in the form

$$\frac{2x^3-3x^2+2x-4}{x^2+4} = 2x - 3 + \frac{-6x+8}{x^2+4}. \tag{1}$$

By applying this process, any improper rational function $P(x)/Q(x)$ can be expressed as the sum of a polynomial and a proper rational function,

$$\frac{P(x)}{Q(x)} = \text{polynomial} + \frac{R(x)}{Q(x)}, \tag{2}$$

where the degree of $R(x)$ is less than the degree of $Q(x)$. In the particular case of (1), this decomposition by means of long division enables us to carry out the integration quite easily, by writing

$$\int \frac{2x^3 - 3x^2 + 2x - 4}{x^2 + 4}\, dx = x^2 - 3x - 6\int \frac{x\, dx}{x^2 + 4} + 8\int \frac{dx}{x^2 + 4}$$

$$= x^2 - 3x - 3 \ln (x^2 + 4) + 4 \tan^{-1} \frac{x}{2} + c.$$

In the general case (2), these remarks tell us that we can restrict our attention to proper rational functions, since the integration of polynomials is always easy. This restriction is not only convenient, but also necessary, because it is *only* to proper rational functions that the following discussions apply.

In elementary algebra we learned how to combine fractions over a common denominator. We must now learn how to reverse this process and split a given fraction into a sum of fractions having simpler denominators. This procedure is called *decomposition into partial fractions.*

Example 1 It is clear that

$$\frac{3}{x - 1} + \frac{2}{x + 3} = \frac{3(x + 3) + 2(x - 1)}{(x - 1)(x + 3)} = \frac{5x + 7}{(x - 1)(x + 3)}. \tag{3}$$

In the reverse process we start with the right side of (3) as our given rational function and seek constants A and B such that

$$\frac{5x + 7}{(x - 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 3}. \tag{4}$$

(For the sake of understanding the method, let us pretend for a moment that we don't know that $A = 3$ and $B = 2$ will work.) If we clear fractions in (4) by multiplying through by $(x - 1)(x + 3)$, we get

$$5x + 7 = A(x + 3) + B(x - 1) \tag{5}$$

or

$$5x + 7 = (A + B)x + (3A - B). \tag{6}$$

Since (6) is to be an identity in x, we can find A and B by equating coefficients of like powers of x. This gives a system of two equations in the two unknowns A and B,

$$\begin{cases} A + B = 5 \\ 3A - B = 7, \end{cases} \quad \text{whose solution is} \quad A = 3,\ B = 2.$$

There is another convenient way to find A and B, by using (5) directly. Since (5) must hold for all x, it must hold in particular for $x = 1$ (which removes B) and for $x = -3$ (which removes A). Briefly,

$x = 1$: $\quad 5 + 7 = A(1 + 3) + 0, \quad 4A = 12, \quad A = 3;$

$x = -3$: $\quad -15 + 7 = 0 + B(-3 - 1), \quad -4B = -8, \quad B = 2.$

This method is faster than it looks, and can be carried out by inspection. Whichever method we use to find A and B, (4) becomes

$$\frac{5x + 7}{(x - 1)(x + 3)} = \frac{3}{x - 1} + \frac{2}{x + 3},$$

and this is the partial fractions decomposition of the rational function on the left. Of course, the purpose of this decomposition is to enable us to integrate the given function,

$$\int \frac{5x+7}{(x-1)(x+3)}\,dx = \int \left(\frac{3}{x-1} + \frac{2}{x+3}\right) dx$$
$$= 3 \ln (x-1) + 2 \ln (x+3) + c.$$

The type of expansion used in (4) works in just the same way under more general circumstances, as follows: Let $P(x)/Q(x)$ be a proper rational function whose denominator $Q(x)$ is an nth-degree polynomial. If $Q(x)$ can be factored completely into *distinct linear factors* $x - r_1, x - r_2, \ldots, x - r_n$, then there exist n constants $A_1, A_2, \ldots, A_n$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}. \tag{7}$$

The constants in the numerators can be determined by either of the methods suggested in Example 1; and when this is done, the partial fractions decomposition (7) provides an easy way to integrate the given rational function.

Example 2 Find

$$\int \frac{6x^2+14x-20}{x^3-4x}\,dx.$$

Solution We factor the denominator by writing $x^3 - 4x = x(x^2-4) = x(x+2)(x-2)$. Accordingly, we have a decomposition of the form

$$\frac{6x^2+14x-20}{x^3-4x} = \frac{6x^2+14x-20}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \tag{8}$$

for certain constants A, B, C. To find these constants we clear fractions in (8), which yields

$$6x^2 + 14x - 20 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2).$$

By setting $x = 0, -2, 2$ (this is the second method in Example 1), we easily see that $A = 5$, $B = -3$, $C = 4$, so (8) becomes

$$\frac{6x^2+14x-20}{x^3-4x} = \frac{5}{x} - \frac{3}{x+2} + \frac{4}{x-2}.$$

We therefore have

$$\int \frac{6x^2+14x-20}{x^3-4x}\,dx = 5 \ln x - 3 \ln (x+2) + 4 \ln (x-2) + c.$$

In theory, every polynomial $Q(x)$ with real coefficients can be factored completely into real linear and quadratic factors, some of which may be repeated.* In practice, this factorization is hard to carry out for polynomials of degree 3 or more, except in special cases. Nevertheless, let us assume this has been done, and let us see how the decomposition (7) must be altered to take account of the most general circumstances that can arise.

*This statement is a consequence of the *Fundamental Theorem of Algebra,* which is discussed in Section 14.8.

If a linear factor $x - r$ occurs with multiplicity m, then the corresponding term $A/(x - r)$ in the decomposition (7) must be replaced by a sum of the form

$$\frac{B_1}{x-r} + \frac{B_2}{(x-r)^2} + \cdots + \frac{B_m}{(x-r)^m}.$$

A quadratic factor $x^2 + bx + c$ of multiplicity 1 gives rise to a single term

$$\frac{Ax+B}{x^2+bx+c},$$

and if this quadratic factor occurs with multiplicity m, then it gives rise to a sum of the form

$$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(x^2+bx+c)^m}.$$

This is the whole story, and the theory guarantees that every proper rational function can be expanded into a sum of partial fractions in the manner described above.*

Example 3 Find

$$\int \frac{3x^3 - 4x^2 - 3x + 2}{x^4 - x^2}\,dx.$$

Solution We have

$$\begin{aligned}\frac{3x^3 - 4x^2 - 3x + 2}{x^4 - x^2} &= \frac{3x^3 - 4x^2 - 3x + 2}{x^2(x+1)(x-1)}\\ &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}.\end{aligned}$$

Clearing fractions gives the identity

$$3x^3 - 4x^2 - 3x + 2 = Ax(x+1)(x-1) + B(x+1)(x-1) + Cx^2(x-1) + Dx^2(x+1).$$

Now put

$x = 0$:	$2 = -B,$	$B = -2;$
$x = 1$:	$-2 = 2D,$	$D = -1;$
$x = -1$:	$-2 = -2C,$	$C = 1.$

Equating coefficients of x^3 gives

$$3 = A + C + D, \qquad \text{so} \qquad A = 3.$$

Our partial fractions decomposition is therefore

$$\frac{3x^3 - 4x^2 - 3x + 2}{x^4 - x^2} = \frac{3}{x} - \frac{2}{x^2} + \frac{1}{x+1} - \frac{1}{x-1},$$

so

$$\int \frac{3x^3 - 4x^2 - 3x + 2}{x^4 - x^2}\,dx = 3\ln x + \frac{2}{x} + \ln(x+1) - \ln(x-1) + c.$$

*This statement is called the *Partial Fractions Theorem*; it is proved in Appendix A.11. Students will notice that the above description of the partial fractions decomposition assumes that the highest power of x in $Q(x)$ has coefficient 1; this can always be arranged by a minor algebraic adjustment.

Example 4 Find

$$\int \frac{2x^3 + x^2 + 2x - 1}{x^4 - 1}\, dx.$$

Solution We have

$$\frac{2x^3 + x^2 + 2x - 1}{x^4 - 1} = \frac{2x^3 + x^2 + 2x - 1}{(x+1)(x-1)(x^2+1)}$$
$$= \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx + D}{x^2 + 1},$$

so

$$2x^3 + x^2 + 2x - 1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + Cx(x^2-1) + D(x^2-1).$$

Now put

$x = 1$:	$4 = 4B$,	$B = 1$;
$x = -1$:	$-4 = -4A$,	$A = 1$;
$x = 0$:	$-1 = -A + B - D$,	$D = 1$.

Equating coefficients of x^3 gives

$$2 = A + B + C, \qquad \text{so} \qquad C = 0.$$

Our partial fractions decomposition is therefore

$$\frac{2x^3 + x^2 + 2x - 1}{x^4 - 1} = \frac{1}{x+1} + \frac{1}{x-1} + \frac{1}{x^2+1},$$

so

$$\int \frac{2x^3 + x^2 + 2x - 1}{x^4 - 1}\, dx = \ln(x+1) + \ln(x-1) + \tan^{-1} x + c.$$

As a final comment, we point out that all the partial fractions that can possibly arise have the form

$$\frac{A}{(x-r)^n} \quad \text{or} \quad \frac{Ax + B}{(x^2 + bx + c)^n}, \qquad n = 1, 2, 3, \ldots.$$

Functions of the first type can be integrated by using the substitution $u = x - r$, and it is clear that the results are always logarithms or rational functions. A function of the second type in which the quadratic polynomial $x^2 + bx + c$ has no real linear factors, that is, in which the roots of $x^2 + bx + c = 0$ are imaginary, can be integrated by completing the square and making a suitable substitution. When this is done, we get integrals of the form

$$\int \frac{u\, du}{(u^2 + k^2)^n}, \qquad \int \frac{du}{(u^2 + k^2)^n}.$$

The first of these is $\frac{1}{2} \ln(u^2 + k^2)$ if $n = 1$, and $(u^2 + k^2)^{1-n}/2(1-n)$ if $n > 1$. When $n = 1$, the second integral is calculated by the formula

$$\int \frac{du}{u^2 + k^2} = \frac{1}{k} \tan^{-1} \frac{u}{k}.$$

The case $n > 1$ can be reduced to the case $n = 1$ by repeated application of the *reduction formula*

$$\int \frac{du}{(u^2 + k^2)^n} = \frac{1}{2k^2(n-1)} \cdot \frac{u}{(u^2 + k^2)^{n-1}} + \frac{2n-3}{2k^2(n-1)} \int \frac{du}{(u^2 + k^2)^{n-1}}. \tag{9}$$

We state this complicated formula for the sole purpose of showing that the only functions that arise from the indicated reduction procedure are rational functions and inverse tangents. The formula itself can either be verified by differentiation or obtained from scratch by the methods of the next section.

This discussion shows that the integral of every rational function can be expressed in terms of polynomials, rational functions, logarithms, and inverse tangents. The detailed work can be very laborious, but at least the path that must be followed is clearly visible.

PROBLEMS

1 Express each of the following improper rational functions as the sum of a polynomial and a proper rational function, and integrate:

(a) $\dfrac{x^2}{x-1}$; (b) $\dfrac{x^3}{3x+2}$; (c) $\dfrac{x^3}{x^2+1}$;

(d) $\dfrac{x+3}{x+2}$; (e) $\dfrac{x^2-1}{x^2+1}$.

Find each of the following integrals.

2 $\displaystyle\int \frac{12x-17}{(x-1)(x-2)}\,dx.$

3 $\displaystyle\int \frac{14x-12}{2x^2-2x-12}\,dx.$

4 $\displaystyle\int \frac{10-2x}{x^2+5x}\,dx.$

5 $\displaystyle\int \frac{2x+21}{x^2-7x}\,dx.$

6 $\displaystyle\int \frac{9x^2-24x+6}{x^3-5x^2+6x}\,dx.$

7 $\displaystyle\int \frac{x^2+46x-48}{x^3+5x^2-24x}\,dx.$

8 $\displaystyle\int \frac{16x^2+3x-7}{x^3-x}\,dx.$

9 $\displaystyle\int \frac{4x^2+11x-117}{x^3+10x^2-39x}\,dx.$

10 $\displaystyle\int \frac{6x^2-9x+9}{x^3-3x^2}\,dx.$

11 $\displaystyle\int \frac{-4x^2-5x-3}{x^3+2x^2+x}\,dx.$

12 $\displaystyle\int \frac{4x^2+2x+4}{x^3+4x}\,dx.$

13 $\displaystyle\int \frac{3x^2-x+4}{x^3+2x^2+2x}\,dx.$

14 $\displaystyle\int \frac{x^4}{x^2+4}\,dx.$

15 $\displaystyle\int \frac{x^4+3x^2-4x+5}{(x-1)^2(x^2+1)}\,dx.$

16 $\displaystyle\int \frac{x^2+2x}{(x+1)^2}\,dx.$

17 $\displaystyle\int \frac{x^2}{x+2}\,dx.$

18 $\displaystyle\int \frac{x+1}{x-1}\,dx.$

19 $\displaystyle\int \frac{x^2+1}{x+2}\,dx.$

20 $\displaystyle\int \frac{x^3-3x^2+2x-3}{x^2+1}\,dx.$

21 $\displaystyle\int \frac{\cos\theta}{\sin^2\theta+3\sin\theta-4}\,d\theta.$

22 $\displaystyle\int \frac{16\sec^2\theta}{\tan^3\theta-4\tan^2\theta}\,d\theta.$

23 $\displaystyle\int \frac{e^x}{e^{2x}-4}\,dx.$

24 $\displaystyle\int \frac{1}{1+e^x}\,dx.$

25 Use partial fractions to obtain the formula

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x}.$$

Also calculate this integral by trigonometric substitution, and verify that the two answers agree.

26 Find

(a) $\displaystyle\int \frac{3\sin\theta\,d\theta}{\cos^2\theta-\cos\theta-2}$; (b) $\displaystyle\int \frac{5e^t\,dt}{e^{2t}+e^t-6}$.

27 In Problem 14 of Section 8.5 it is stated that the differential equation

$$\frac{dx}{dt} = kab(A-x)(B-x), \qquad A \neq B,$$

has

$$\frac{B(A-x)}{A(B-x)} = e^{kab(A-B)t}$$

as a solution for which $x = 0$ when $t = 0$. Derive this solution by using partial fractions.

28 Verify the reduction formula (9) by differentiating the first term on the right.

29 Suppose that a given population can be divided into two

groups: those who have a certain infectious disease, and those who do not have it but can catch it by having contact with an infected person. If x and y are the proportions of infected and uninfected people, then $x + y = 1$. Assume (1) that the disease spreads by the contacts just mentioned between sick people and well people, (2) that the rate of spread dx/dt is proportional to the number of such contacts, and (3) that the two such contacts, and (3) that the two groups mingle freely with each other, so that the number of contacts is jointly proportional to x and y. If $x = x_0$ when $t = 0$, find x as a function of t, sketch the graph, and use this function to show that ultimately the disease will spread through the entire population.
When the formula for the derivative of a product (the

10.7 INTEGRATION BY PARTS

When the formula for the derivative of a product (the product rule) is written in the notation of differentials, it is

$$d(uv) = u\,dv + v\,du \qquad \text{or} \qquad u\,dv = d(uv) - v\,du,$$

and by integrating we obtain

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

This formula provides a method of finding $\int u\,dv$ if the second integral $\int v\,du$ is easier to calculate than the first. The method is called *integration by parts,* and it often works when all other methods fail.

Example 1 Find $\int x \cos x\,dx$.

Solution If we put

$$u = x, \qquad dv = \cos x\,dx,$$

then

$$du = dx, \qquad v = \sin x,$$

and (1) gives

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx.$$

This is good luck, because the integral on the right is easy. We therefore have

$$\int x \cos x\,dx = x \sin x + \cos x + c.$$

It is worth noticing that in this example we could have chosen u and dv differently. If we put

$$u = \cos x, \qquad dv = x\,dx,$$

then

$$du = -\sin x\,dx, \qquad v = \tfrac{1}{2}x^2,$$

and (1) gives

$$\int x \cos x\,dx = \tfrac{1}{2}x^2 \cos x + \tfrac{1}{2}\int x^2 \sin x\,dx.$$

This equation is true, but it is completely worthless as a means of solving our problem, because the second integral is harder than the first. We urge students to learn from experience, and to use trial and error as intelligently as possible in choosing u and dv. Also, students should feel free to abandon a choice that doesn't seem to work, and quickly go on to another choice that offers more hope of success.

The method of integration by parts applies particularly well to products of different types of functions, like $x \cos x$ in Example 1, which is a product of a polynomial and a trigonometric function. In using the method, the given differential must be thought of as a product $u \cdot dv$. The part called dv must be something we can integrate, and the part called u should usually be something that is simplified by differentiation, as in our next example.

Example 2 Find $\int \ln x \, dx$.

Solution Here our only choice is

$$u = \ln x, \qquad dv = dx,$$

so

$$u = \frac{dx}{x}, \qquad v = x,$$

and we have

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x} = x \ln x - x + c.$$

In some cases it is necessary to carry out two or more integrations by parts in succession.

Example 3 Find $\int x^2 e^x \, dx$.

Solution If we put

$$u = x^2, \qquad dv = e^x \, dx,$$

then

$$du = 2x \, dx, \qquad v = e^x,$$

and (1) gives

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx. \tag{2}$$

Here the second integral is easier than the first, so we are encouraged to continue in the same way. When the second integral is integrated by parts with

$$u = x, \qquad dv = e^x \, dx,$$

so that

$$du = dx, \qquad v = e^x,$$

then we get

$$\int x e^x \, dx = x e^x - \int e^x \, dx$$

$$= x e^x - e^x.$$

When this is inserted in (2), our final result is

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + c.$$

It sometimes happens that the integral we start with appears a second time during the integration by parts, in which case it is often possible to solve for this integral by elementary algebra.

Example 4 Find $\int e^x \cos x\,dx$.

Solution For convenience we denote this integral by J. If we put

$$u = e^x, \qquad dv = \cos x\,dx,$$

then

$$du = e^x\,dx, \qquad v = \sin x,$$

and (1) yields

$$J = e^x \sin x - \int e^x \sin x\,dx. \tag{3}$$

Now we come to the interesting part of this problem. Even though the new integral is no easier than the old, it turns out to be fruitful to apply the same method again to the new integral. Thus, we put

$$u = e^x, \qquad dv = \sin x\,dx,$$

so that

$$du = e^x\,dx, \qquad v = -\cos x,$$

and obtain

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx. \tag{4}$$

The integral on the right is J again, so (4) can be written

$$\int e^x \sin x\,dx = -e^x \cos x + J. \tag{5}$$

In spite of appearances, we are not going in a circle, because substituting (5) in (3) gives

$$J = e^x \sin x + e^x \cos x - J.$$

It is now easy to solve for J by writing

$$2J = e^x \sin x + e^x \cos x \qquad \text{or} \qquad J = \tfrac{1}{2}(e^x \sin x + e^x \cos x),$$

and all that remains is to insert the constant of integration:

$$\int e^x \cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + c.$$

The method of this example is often used to make an integral depend on a simpler integral of the same type, and thus to obtain a convenient *reduction formula,* by repeated use of which the given integral can easily be calculated.

Example 5 Find a reduction formula for $J_n = \int \sin^n x\, dx$.

Solution We integrate by parts with

$$u = \sin^{n-1} x, \qquad dv = \sin x\, dx,$$

so that

$$du = (n-1)\sin^{n-2} x \cos x\, dx, \qquad v = -\cos x,$$

and therefore

$$\begin{aligned} J_n &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\, J_{n-2} - (n-1)\, J_n. \end{aligned}$$

We now transpose the term involving J_n and obtain

$$nJ_n = -\sin^{n-1} x \cos x + (n-1)J_{n-2},$$

so that

$$J_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} J_{n-2},$$

or equivalently,

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx. \tag{6}$$

The reduction formula (6) allows us to reduce the exponent on $\sin x$ by 2. By repeated application of this formula we can therefore ultimately reduce J_n to J_0 or J_1, according as n is even or odd. But both of these are easy:

$$J_0 = \int \sin^0 x\, dx = \int dx = x \qquad \text{and} \qquad J_1 = \int \sin x\, dx = -\cos x.$$

For example, with $n = 4$ we get

$$\int \sin^4 x\, dx = -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}\int \sin^2 x\, dx,$$

and with $n = 2$,

$$\begin{aligned} \int \sin^2 x\, dx &= -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}\int dx \\ &= -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x. \end{aligned}$$

Therefore,

$$\begin{aligned} \int \sin^4 x\, dx &= -\tfrac{1}{4}\sin^3 x \cos x + \tfrac{3}{4}(-\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x) \\ &= -\tfrac{1}{4}\sin^3 x \cos x - \tfrac{3}{8}\sin x \cos x + \tfrac{3}{8}x + c. \end{aligned}$$

The same result can be achieved by earlier techniques depending on repeated use of the half-angle formulas, but our present methods are more efficient for large exponents. In our next example we illustrate another way in which the reduction formula (6) can be used.

Example 6 Calculate

$$\int_0^{\pi/2} \sin^8 x\, dx.$$

Solution For convenience we write

$$I_n = \int_0^{\pi/2} \sin^n x\, dx.$$

By formula (6) we have

$$I_n = -\frac{1}{n} \sin^{n-1} x \cos x \Big]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx,$$

so

$$I_n = \frac{n-1}{n} I_{n-2}.$$

We apply this formula with $n = 8$, then repeat with $n = 6$, $n = 4$, $n = 2$:

$$I_8 = \tfrac{7}{8} I_6 = \tfrac{7}{8} \cdot \tfrac{5}{6} I_4 = \tfrac{7}{8} \cdot \tfrac{5}{6} \cdot \tfrac{3}{4} I_2 = \tfrac{7}{8} \cdot \tfrac{5}{6} \cdot \tfrac{3}{4} \cdot \tfrac{1}{2} I_0.$$

Therefore

$$\int_0^{\pi/2} \sin^8 x\, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35\pi}{256}.$$

Remark 1 The reduction formula (6) can also be used to establish one of the most fascinating formulas of mathematics, *Wallis's infinite product* for $\pi/2$:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots.$$

For the details of the proof, see Appendix 2 at the end of the chapter.

Remark 2 In Section 9.5 we stated *Leibniz's formula* for $\pi/4$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

For students who are interested in little-known corners of the history of mathematics, we describe in Appendix 3 at the end of the chapter how Leibniz himself discovered his formula by a very ingenious application of integration by parts.

At this point we have described all the standard methods of integration that the student is expected to be acquainted with. A few additional techniques of minor importance remain, and two of these are briefly sketched in the problems at the end of Appendix A.9; but for most practical purposes we have reached the end of this particular road.

PROBLEMS

Find each of the following integrals by the method of integration by parts.

1 $\int x \ln x \, dx$.
2 $\int \tan^{-1} x \, dx$.
3 $\int x \tan^{-1} x \, dx$.
4 $\int xe^{ax} \, dx$.
5 $\int e^x \sin x \, dx$.
6 $\int e^{ax} \cos bx \, dx$.
7 $\int \sqrt{1 - x^2} \, dx$.
8 $\int \sin^{-1} x \, dx$.
9 $\int x \sin^{-1} x \, dx$.
10 $\int_0^{\pi/2} x \sin x \, dx$.
11 $\int x \cos (3x - 2) \, dx$.
12 $\int \frac{\tan^{-1} x}{x^2} \, dx$.
13 $\int x \sec^2 x \, dx$.
14 $\int \sin (\ln x) \, dx$.
15 $\int \ln (a^2 + x^2) \, dx$.
16 $\int x^2 \ln (x + 1) \, dx$.
17 $\int \frac{\ln x}{x} \, dx$.
18 $\int (\ln x)^2 \, dx$.

19 The region under the curve $y = \cos x$ between $x = 0$ and $x = \pi/2$ is revolved about the y-axis. Find the volume of the resulting solid.

20 Find $\int (\sin^{-1} x)^2 \, dx$. Hint: Make the substitution $y = \sin^{-1} x$.

21 If $P(x)$ is a polynomial, show that

$$\int P(x)e^x \, dx = (P - P' + P'' - P''' + \cdots)e^x.$$

In the next two problems, derive the given reduction formula and apply it to the indicated special case(s).

22 (a) $$\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

(b) $\int_0^{\pi/2} \cos^7 x \, dx$.

(c) $\int_0^{\pi/2} \cos^8 x \, dx$.

23 (a) $\int (\ln x)^n \, dx = x(\ln x)^n - n\int (\ln x)^{n-1} \, dx$.

(b) $\int (\ln x)^5 \, dx$.

24 The region under the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is revolved about the y-axis. Find the volume of the resulting solid (a) by the shell method; and (b) by the washer method.

25 The curve in Problem 24 is revolved about the x-axis. Find the area of the resulting surface of revolution.

26 (The volcanic ash problem) When a volcano erupts, the cloud of ejected ash gradually settles onto the surface of the nearby land. The depth of the deposited layer of ash decreases with distance from the volcano. Assume that the depth of the ash r feet from the volcano is ae^{-br} feet, where a and b are positive constants.

(a) Find the total volume of ash that falls within a distance c of the volcano. Hint: What is the element of volume dV of ash that falls on a narrow ring of width dr and inner radius r centered on the volcano?

(b) What is the limit of this volume as $c \to \infty$?

10.8 A MIXED BAG. STRATEGY FOR DEALING WITH INTEGRALS OF MISCELLANEOUS TYPES

As the student understands by now, differentiation is straightforward but integration is not. In finding the derivative of a function it is obvious which formula must be applied. But it may not be at all obvious which method should be used to integrate a given function.

Since the problems in each section of this chapter have focused on the methods of that section, it has usually been clear what method to use on a given integral. Generally speaking, the methods at our disposal now are direct substitution, trigonometric substitution, partial fractions, and parts. But what if an integral is met out of context, with no obvious clue as to how to work it out? In this section we try to suggest a strategy for this common situation.

An essential prerequisite is a knowledge of the basic integration formulas. For the sake of emphasis, we repeat the list given in Section 10.1, together with three additional formulas arising from our work in this chapter. As we pointed out earlier, the first 15 formulas should be memorized, and we hope students will take our advice seriously this time. It is useful to know them all, but the last three (marked with an asterisk) need not be memorized since they are easy to derive, as follows. Formulas 16 and 17 are immediate from the simple partial fractions decompositions

$$\frac{1}{x^2 - a^2} = \frac{1}{(x+a)(x-a)} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

and

$$\frac{1}{a^2 - x^2} = \frac{1}{(a+x)(a-x)} = \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right].$$

These decompositions can easily be understood by mentally recombining the terms in brackets with the aid of a common denominator; we then see directly what the constant factor outside the brackets must be. Formula 18 is almost immediate from the trigonometric substitutions $x = a \tan \theta$ and $x = a \sec \theta$, respectively. In this list of formulas we use x instead of u as the variable of integration—since the usefulness of the u-notation is now thoroughly familiar to us—and for the sake of simplicity we omit the constant of integration.

1 $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1} \qquad (n \neq -1).$

2 $\displaystyle\int \frac{dx}{x} = \ln x.$

3 $\displaystyle\int e^x \, dx = e^x.$

4 $\displaystyle\int \cos x \, dx = \sin x.$

5 $\displaystyle\int \sin x \, dx = -\cos x.$

6 $\displaystyle\int \sec^2 x \, dx = \tan x.$

7 $\displaystyle\int \csc^2 x \, dx = -\cot x.$

8 $\displaystyle\int \sec x \tan x \, dx = \sec x.$

9 $\displaystyle\int \csc x \cot x \, dx = -\csc x.$

10 $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}.$

11 $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}.$

12 $\displaystyle\int \tan x \, dx = -\ln (\cos x).$

13 $\displaystyle\int \cot x \, dx = \ln (\sin x).$

14 $\displaystyle\int \sec x \, dx = \ln (\sec x + \tan x).$

15 $\displaystyle\int \csc x \, dx = -\ln (\csc x + \cot x).$

*16 $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left(\frac{x-a}{x+a}\right).$

*17 $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left(\frac{a+x}{a-x}\right).$

*18 $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln (x + \sqrt{x^2 \pm a^2}).$

These formulas constitute our arsenal of weapons for attacking integrals, and it is up to us to decide which to use in a particular case. If we do not see what to do immediately, the following strategy may be helpful.

STRATEGY FOR INTEGRATION

1 *Simplify the integrand.* The use of algebraic or trigonometric identities will sometimes simplify the integrand and make a method of integration obvious. For example:

$$\int \sqrt{x}(\sqrt{x} + \sqrt[3]{x})\, dx = \int (x + x^{5/6})\, dx;$$

$$\int (\sin x + \cos x)^2\, dx = \int (\sin^2 x + 2 \sin x \cos x + \cos^2 x)\, dx$$

$$= \int (1 + 2 \sin x \cos x)\, dx;$$

$$\int \frac{1 - \tan^2 x}{\sec^2 x}\, dx = \int (1 - \tan^2 x) \cos^2 x\, dx$$

$$= \int \left(1 - \frac{\sin^2 x}{\cos^2 x}\right) \cos^2 x\, dx$$

$$= \int (\cos^2 x - \sin^2 x)\, dx = \int \cos 2x\, dx.$$

In the second problem, if we fail to notice that $\sin^2 x + \cos^2 x = 1$, and instead integrate $\sin^2 x$ and $\cos^2 x$ separately, then we can still solve the problem, but we have missed an opportunity to do things the easy way. A similar remark applies to the third problem, with its use of the double-angle formula for the cosine.

2 *Look for an obvious substitution.* Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)\, dx$ is also present, apart from a constant factor. For example, in

$$\int \frac{x\, dx}{4 - x^2}$$

we notice that if $u = 4 - x^2$, then $du = -2x\, dx$ and $x\, dx = -\frac{1}{2}\, du$. It is therefore much simpler to use this substitution than to use partial fractions or the trigonometric substitution $x = 2 \sin \theta$, each of which also works but takes longer to carry out.

3 *Classify the integrand.* This is the heart of the matter. If Steps 1 and 2 have not helped, then we turn to a more careful examination of the form of the integrand $f(x)$.

(a) If $f(x)$ is (or can be written as) a product of powers of $\sin x$ and $\cos x$, or $\tan x$ and $\sec x$, or $\cot x$ and $\csc x$, then the methods of Section 10.3 can be used.*

*A special method for integrating *any* rational function of $\sin x$ and $\cos x$ is described in Appendix A.9 at the end of the book. This method will not be needed for any of the review problems at the end of this section.

(b) If $f(x)$ involves $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 \pm a^2}$, or powers of these expressions, use the trigonometric substitutions of Section 10.4.
(c) If $f(x)$ is a rational function, use partial fractions as explained in Section 10.6—unless there is a better way for a particular integral.
(d) If $f(x)$ is a product of functions of different types, try integration by parts. As we have seen in Section 10.7, this method also works for many individual inverse functions like $\ln x$, $\sin^{-1} x$, and $\tan^{-1} x$.
(e) Be observant, thoughtful, flexible and persistent—all of which are of course easier said than done. If a method doesn't work, be ready to try another. Sometimes several methods work. Keep your options open and do things the easy way—if any. And remember that doing a problem more than one way is a good learning experience.

Our purpose in the following examples is to try to suggest possible lines of attack by "thinking out loud." We are interested mainly in brainstorming these integrals, and in most cases we will not work out all the details to the final answer.

Example 1 $\displaystyle\int \frac{x^2}{x^6 - 1}\,dx.$

Comments Since the integrand is a rational function, partial fractions will work. This requires factoring $x^6 - 1$ into $(x^3 + 1)(x^3 - 1) = (x + 1)(x^2 - x + 1)\cdot(x - 1)(x^2 + x + 1)$ and then finding constants A, B, C, D, E, F such that

$$\frac{x^2}{x^6 - 1} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1} + \frac{D}{x - 1} + \frac{Ex + F}{x^2 + x + 1}.$$

We can do this if we must, but actually carrying out this work is not an attractive prospect.

Let us probe in a different direction. A much more promising method is to notice that x^6 is the square of x^3 and that the numerator of the integrand is almost the derivative of x^3. Accordingly, if we put $u = x^3$, then $du = 3x^2\,dx$, $x^2\,dx = \frac{1}{3}\,du$, and the integral becomes

$$\frac{1}{3}\int \frac{du}{u^2 - 1} = \frac{1}{6}\ln\left(\frac{u - 1}{u + 1}\right) = \frac{1}{6}\ln\left(\frac{x^3 - 1}{x^3 + 1}\right),$$

by formula 16.

Example 2 $\displaystyle\int \frac{x^2}{1 + x^2}\,dx.$

Comments The trigonometric substitution $x = \tan\theta$ will work. Partial fractions will also work, but since the integrand is an improper rational function, we must begin with long division. However, an easier way to accomplish the result without actually carrying out the long division is simply to add and subtract 1 in the numerator,

$$\int \frac{x^2}{1 + x^2}\,dx = \int \left(\frac{x^2 + 1 - 1}{1 + x^2}\right) dx = \int \left(1 - \frac{1}{1 + x^2}\right) dx$$
$$= x - \tan^{-1} x.$$

Example 3 $\int \frac{e^{2x}\,dx}{e^x - 1}$.

Comments We begin by noticing that $e^{2x}\,dx = e^x(e^x\,dx) = e^x\,d(e^x)$. This suggests that we put $u = e^x$, so that $e^{2x}\,dx = u\,du$ and the integral can be written

$$\int \frac{u\,du}{u-1} = \int \frac{u-1+1}{u-1}\,du = \int \left(1 + \frac{1}{u-1}\right) du$$
$$= u + \ln(u-1) = e^x + \ln(e^x - 1).$$

By subtracting and adding 1 here we employ a slight variation of the idea used in Example 2.

Example 4 $\int \frac{4x+1}{1+x^2}\,dx$.

Comments The numerator is nearly (but not quite) the derivative of the denominator. This suggests that we break the integrand into a sum and rearrange the constants to achieve this desirable condition:

$$\int \frac{4x+1}{1+x^2}\,dx = \int \left(2 \cdot \frac{2x}{1+x^2} + \frac{1}{1+x^2}\right) dx$$
$$= 2\int \frac{2x\,dx}{1+x^2} + \int \frac{dx}{1+x^2} = 2\ln(1+x^2) + \tan^{-1} x.$$

Example 5 $\int \frac{2x+6}{x^2+7x+10}\,dx$.

Comments In Example 4 we arranged part of the numerator to be the derivative of the denominator. A similar purpose here suggests that we write

$$\int \frac{2x+6}{x^2+7x+10}\,dx = \int \frac{(2x+7)-1}{x^2+7x+10}\,dx$$
$$= \int \frac{(2x+7)\,dx}{x^2+7x+10} - \int \frac{dx}{x^2+7x+10}.$$

The first of these integrals has been arranged to be $\ln(x^2+7x+10)$, and we can easily work out the second by factoring the denominator into $(x+2)(x+5)$ and using partial fractions.

Example 6 $\int \frac{x^5\,dx}{(1+x^2)^4}$.

Comments The trigonometric substitution $x = \tan\theta$ will work. Partial fractions will also work, but if we try this there will be eight unknown constants to find. We hope for something better.

Let us try the substitution $u = 1 + x^2$. Our only reason for this is that it simplifies the denominator to u^4. Then $du = 2x\,dx$, and we have

$$\int \frac{x^5\,dx}{(1+x^2)^4} = \int \frac{(x^2)^2\,(x\,dx)}{(1+x^2)^4} = \frac{1}{2}\int \frac{(u-1)^2\,du}{u^4}$$
$$= \frac{1}{2}\int \frac{u^2-2u+1}{u^4}\,du = \frac{1}{2}\int (u^{-2}-2u^{-3}+u^{-4})\,du,$$

which is easy.

Example 7 $\displaystyle\int \frac{dx}{x(\ln x)^2}$.

Comments We notice at once that the differential of $\ln x$ is dx/x. We therefore put $u = \ln x$, so that $du = dx/x$ and

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\ln x}.$$

Example 8 $\displaystyle\int \frac{x\,dx}{\sqrt[3]{x+1}}$.

Comments This requires a so-called *rationalizing substitution,* that is, one that eliminates the radical. We put $u = \sqrt[3]{x+1}$, so that $u^3 = x+1$, $3u^2\,du = dx$, and $x = u^3 - 1$. We can now write

$$\int \frac{x\,dx}{\sqrt[3]{x+1}} = \int \frac{(u^3-1)3u^2\,du}{u} = \int (3u^4 - 3u)\,du,$$

which is easy.

Example 9 $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$.

Comments The rationalizing substitution

$$u = \sqrt{\frac{1+x}{1-x}}$$

will work here, but the result is a messy rational function. A better idea is to multiply both numerator and denominator by $\sqrt{1+x}$, which gives

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{1+x}{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1} x - \sqrt{1-x^2}.$$

Example 10 $\displaystyle\int \frac{1}{1+\cos x}\,dx$.

Comments This time we multiply both numerator and denominator by $1 - \cos x$ to obtain a somewhat different application of the idea in Example 9:

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}\,dx = \int \frac{1-\cos x}{1-\cos^2 x}\,dx$$
$$= \int \frac{1-\cos x}{\sin^2 x}\,dx = \int \csc^2 x\,dx - \int \frac{\cos x\,dx}{\sin^2 x}$$
$$= -\cot x + \frac{1}{\sin x}.$$

Example 11 $\int e^{\sqrt{x}}\,dx.$

Comments It is natural to try the substitution $u = \sqrt{x}$, even though we have no idea what is likely to happen. Then $u^2 = x$, $2u\,du = dx$, and we have

$$\int e^{\sqrt{x}}\,dx = \int 2\,ue^u\,du.$$

This integral is now an obvious candidate for integration by parts.

The following list of problems contains integrals of all the types we have encountered, arranged in random order so that students can test their diagnostic powers.

PROBLEMS

Find the following integrals.

1 $\int \frac{x\,dx}{\sqrt{1-x^2}}$.

2 $\int x^4 \ln x\,dx$.

3 $\int \sin^2 x \cos^5 x\,dx$.

4 $\int \frac{dx}{x^3+4x}$.

5 $\int \frac{\sqrt{1+\ln x}}{x \ln x}\,dx$.

6 $\int (e^{3x})^4 e^x\,dx$.

7 $\int \sin \sqrt{x}\,dx$.

8 $\int \frac{x^3}{x^4-1}\,dx$.

9 $\int \cos x \tan x\,dx$.

10 $\int \frac{\cos x}{1+\sin^2 x}\,dx$.

11 $\int x \sin^2 x\,dx$.

12 $\int \frac{\ln x + \sqrt{x}}{x}\,dx$.

13 $\int \frac{e^{2x}}{1+e^x}\,dx$.

14 $\int \frac{\ln (x+1)}{x^2}\,dx$.

15 $\int \frac{x^2\,dx}{\sqrt[3]{x-1}}$.

16 $\int \sin x \cos (\cos x)\,dx$.

17 $\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\,dx$.

18 $\int \sec^4 x\,dx$.

19 $\int \frac{3x+5}{x-2}\,dx$.

20 $\int (1+\sqrt{x})^8\,dx$.

21 $\int \frac{\sqrt{4-x^2}}{x}\,dx$.

22 $\int \frac{dx}{e^{2x}+5e^x}$.

23 $\int x^5 e^{-x^3}\,dx$.

24 $\int (e^x+1)^2\,dx$.

25 $\int \frac{x}{(x+3)^2}\,dx$.

26 $\int x\sqrt[3]{x+5}\,dx$.

27 $\int \frac{x}{x^4+2x^2+10}\,dx$.

28 $\int x^2\sqrt{x^3-4}\,dx$.

29 $\int \frac{x \ln x}{\sqrt{x^2-1}}\,dx$.

30 $\int \frac{\sin 2x}{\sqrt{4-\cos^4 x}}\,dx$.

31 $\int x^2 \sin x^3\,dx$.

32 $\int x \sec x \tan x\,dx$.

33 $\int \frac{x}{(x^2+1)(x^2+4)}\,dx$.

34 $\int \frac{dx}{x\sqrt{2x-16}}$.

35 $\int \frac{x^2\,dx}{(x-1)^3}$.

36 $\int x^3 \ln x\,dx$.

37 $\int \tan^3 x \sec^4 x\,dx$.

38 $\int \left(e^x - \frac{1}{e^x}\right)^2 dx$.

39 $\int \frac{x}{1-x^2+\sqrt{1-x^2}}\,dx$.

40 $\int x^3 e^{-2x}\,dx$.

41 $\int \sin^2 x \cos^4 x\,dx$.

42 $\int \frac{x\,dx}{\sqrt{1-4x^2}}$.

43 $\int \frac{\sqrt{x-1}}{x+3}\,dx$

44 $\int \frac{x^3}{16+x^8}\,dx$.

45 $\int \ln (x^2+3)\,dx$.

46 $\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$.

47 $\int e^{5x} \cos 3x\, dx.$

48 $\int \frac{x+1}{x^2-2x+2}\, dx.$

49 $\int \frac{x\, dx}{x^4-2x^2-3}.$

50 $\int e^{x+e^x}\, dx.$

51 $\int \frac{x^4+1}{x^5+5x+3}\, dx.$

52 $\int x \sin^{-1}(x^2)\, dx.$

53 $\int \frac{dx}{x+7+5\sqrt{x+1}}.$

54 $\int \frac{dx}{x^3+x^2+x+1}.$

55 $\int \frac{\sin x\, dx}{1+3\cos^2 x}.$

56 $\int \frac{2x+3}{x^2+1}\, dx.$

57 $\int \frac{x^3}{(x+1)^8}\, dx.$

58 $\int \sin(\ln x)\, dx.$

59 $\int \tan^6 x\, dx.$

60 $\int \frac{\sin x+\cos x}{\sin x-\cos x}\, dx.$

61 $\int \frac{x}{x^2+5x+6}\, dx.$

62 $\int \frac{dx}{\sqrt{1-4x^2}}.$

63 $\int \ln \sqrt{2x-1}\, dx.$

64 $\int \frac{4\, dx}{x^2+4x+20}.$

65 $\int \sqrt{\frac{1+x}{x-1}}\, dx.$

66 $\int \frac{x}{\sqrt{16-x^4}}\, dx.$

67 $\int \sin^2 5x \cos^2 5x\, dx.$

68 $\int \frac{dx}{x^2+5x-6}.$

69 $\int \cot x \ln(\sin x)\, dx.$

70 $\int \frac{dx}{e^{3x}-e^x}.$

71 $\int \cot^3 2x \csc^3 2x\, dx.$

72 $\int \frac{e^x+1}{e^x-1}\, dx.$

73 $\int \ln(2x+x^2)\, dx.$

74 $\int \frac{dx}{\sin 4x}.$

75 $\int \sqrt[3]{x}(1-\sqrt{x})\, dx.$

76 $\int \frac{e^x\, dx}{e^{2x}-1}.$

77 $\int \ln(1+x^2)\, dx.$

78 $\int \frac{x^4}{(x^5+1)^3}\, dx.$

79 $\int x \tan^2 x\, dx.$

80 $\int \frac{\tan^{-1} 2x}{1+4x^2}\, dx.$

81 $\int \sec^7 x \tan x\, dx.$

82 $\int \frac{dx}{x^2+5x+6}.$

83 $\int x \sin^{-1} x\, dx.$

84 $\int \frac{1+\cos^2 x}{1-\cos^2 x}\, dx.$

85 $\int \frac{x^3}{1+x^8}\, dx.$

86 $\int \frac{\tan^{-1} x}{x^2}\, dx.$

87 $\int \frac{\sec^2 x\, dx}{\sqrt{\sec^2 x-1}}.$

88 $\int \frac{dx}{1+2e^x-e^{-x}}.$

89 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx.$

90 $\int \frac{\sqrt{x^2+9}}{x}\, dx.$

91 $\int \frac{\ln x^{10}}{x}\, dx.$

92 $\int \frac{dx}{\cos 5x}.$

93 $\int \frac{\sqrt{x-2}}{x+2}\, dx.$

94 $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\, dx.$

95 $\int \frac{dx}{\sin^2 x}.$

96 $\int \frac{dx}{\sqrt{9x^2+12x-5}}.$

97 $\int \sqrt{(1+3x)(1-3x)}\, dx.$

98 $\int \sin^3 x \cos^2 x\, dx.$

99 $\int \frac{2x+5}{x^2+5x+6}\, dx.$

100 $\int \sin^2 x \cos^3 x\, dx.$

101 $\int \frac{x}{x^4-2x^2-3}\, dx.$

102 $\int \sqrt{1+\sqrt{1+\sqrt{x}}}\, dx.$

103 $\int \frac{x^4}{x^3-1}\, dx.$

104 $\int x \sec^2 x\, dx.$

105 $\int \frac{\sin^3 x}{\cos^5 x}\, dx.$

106 $\int \sin 2x \cos x\, dx.$

107 $\int \sin x \cos 2x\, dx.$

108 $\int \frac{4}{x^4-1}\, dx.$

109 $\int \frac{x\, dx}{\sqrt[3]{x-1}}.$

110 $\int \frac{dx}{e^x+5+4e^{-x}}.$

111 $\int \ln(ax+b)\, dx.$

112 $\int \frac{\cos^4 x}{\sin^2 x}\, dx.$

113 $\int e^x \cos(e^x)\, dx.$

114 $\int \frac{dx}{3x^2-13x+4}.$

115 $\int \sqrt{x} \ln x\, dx.$

116 $\int \frac{\cos x}{\sin^2 x-2\sin x+3}\, dx.$

117 $\int (\tan x+\cot x)^2\, dx.$

118 $\int \ln(x\sqrt{x})\, dx.$

119 $\int \frac{32x+80}{(x-1)(x+3)^2}\, dx.$

120 $\int \frac{dx}{a^2+b^2x^2}.$

121 $\int \ln(1-\sqrt{x})\, dx.$

122 $\int \frac{x^2\, dx}{(x^2+1)^3}.$

123 $\int x \tan^{-1}(x-1)\, dx.$

124 $\int \frac{dx}{2x^2-2x+1}.$

125 $\int \frac{-x^2}{\sqrt{1-x^2}}\, dx.$

10.9 NUMERICAL INTEGRATION. SIMPSON'S RULE

From the point of view of the theorist, the main value of calculus is intellectual; it helps us comprehend the underlying connections among natural phenomena. However, anyone who uses calculus as a practical tool in science or engineering must occasionally face the question of how the theory can be applied to yield useful methods for performing actual numerical calculations.

In this section we consider the problem of computing the numerical value of a definite integral

$$\int_a^b f(x)\,dx \tag{1}$$

in decimal form to any desired degree of accuracy. In order to find the value of (1) by using the formula

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{2}$$

we must be able to find an indefinite integral $F(x)$ and we must be able to evaluate it at both $x = a$ and $x = b$. When this is not possible, formula (2) is useless. This approach fails even for such simple-looking integrals as

$$\int_0^{\pi} \sqrt{\sin x}\,dx \quad \text{and} \quad \int_1^5 \frac{e^x}{x}\,dx,$$

because there are no elementary functions whose derivatives are $\sqrt{\sin x}$ and e^x/x (see Appendix A.9).

Our purpose here is to describe two methods of computing the numerical value of (1) as accurately as we wish by simple procedures that can be applied regardless of whether an indefinite integral can be found or not. The formulas we develop use only simple arithmetic and the values of $f(x)$ at a finite number of points in the interval $[a, b]$. In comparison with the use of the approximating sums that are used in defining the integral (see Section 6.4), the formulas of this section are more efficient in the sense that they give much better accuracy for the same amount of computational labor.

THE TRAPEZOIDAL RULE

Let the interval $[a, b]$ be divided into n equal parts by points $x_0, x_1, \ldots, x_n$ from $x_0 = a$ to $x_n = b$. Let $y_0, y_1, \ldots, y_n$ be the corresponding values of $y = f(x)$. We then approximate the area between $y = f(x)$ and the x-axis, for $x_{k-1} \le x \le x_k$, by the trapezoid whose upper edge is the segment joining the points (x_{k-1}, y_{k-1}) and (x_k, y_k) [see Fig. 10.5]. The area of this trapezoid is clearly

$$\tfrac{1}{2}(y_{k-1} + y_k)(x_k - x_{k-1}). \tag{3}$$

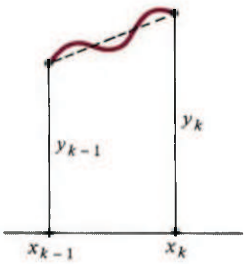

Figure 10.5

If we write

$$\Delta x = x_k - x_{k-1} = \frac{b-a}{n}, \tag{4}$$

then adding the expressions (3) for $k = 1, 2, \ldots, n$ gives the approximation formula

$$\int_a^b f(x)\,dx \cong (\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n)\,\Delta x,$$

because each of the y's except the first and the last occurs twice. This formula is called the *trapezoidal rule.*

Example 1 Use the trapezoidal rule with $n = 4$ to calculate an approximate value for the integral

$$\int_0^1 \sqrt{1 - x^3}\,dx.$$

Here $y = f(x) = \sqrt{1 - x^3}$ and $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $x_4 = 1$. We can compute the y's easily by using a calculator:

$$\begin{aligned} y_0 &= 1, \\ y_1 &= \sqrt{\tfrac{63}{64}} = \sqrt{0.984} = 0.992, \\ y_2 &= \sqrt{\tfrac{7}{8}} = \sqrt{0.875} = 0.935, \\ y_3 &= \sqrt{\tfrac{37}{64}} = \sqrt{0.578} = 0.760, \\ y_4 &= 0. \end{aligned}$$

By the trapezoidal rule, we therefore have

$$\int_0^1 \sqrt{1 - x^3}\, dx \cong \tfrac{1}{4}(0.500 + 0.992 + 0.935 + 0.760 + 0.000) = 0.797.$$

SIMPSON'S RULE*

Our second method is based on a more ingenious device than approximating each small piece of the curve by a line segment; this time we approximate each piece by a portion of a parabola that "fits" the curve in a manner to be described.

Again we divide the interval $[a, b]$ into n equal parts, but now we require that n be an *even* integer. Consider the first three points x_0, x_1, x_2 and the corresponding points on the curve $y = f(x)$, as shown in Fig. 10.6. If these points are not collinear, then there is a unique parabola that has vertical axis and that passes through all three points. To see this, recall that the equation of any parabola with vertical axis has the form $y = P(x)$ where $P(x)$ is a quadratic polynomial, and observe that this polynomial can always be written in the form

$$P(x) = a + b(x - x_1) + c(x - x_1)^2. \tag{5}$$

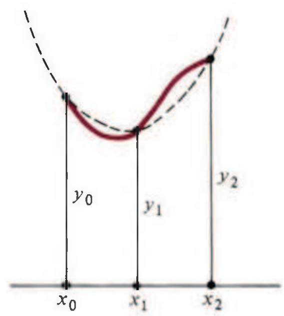

Figure 10.6

We choose the constants a, b, c to make the parabola pass through the three points under consideration, as indicated in the figure. Three conditions are necessary for this:

$$\text{at } x = \begin{cases} x_0, & a + b(x_0 - x_1) + c(x_0 - x_1)^2 = y_0; & (6) \\ x_1, & a = y_1; \\ x_2, & a + b(x_2 - x_1) + c(x_2 - x_1)^2 = y_2. & (7) \end{cases}$$

Equations (6) and (7) can be solved for the constants b and c. However, it is more convenient to use the definition (4) of Δx and the fact that $a = y_1$ to write these equations in the form

$$\begin{aligned} -b\, \Delta x + c\, \Delta x^2 &= y_0 - y_1, \\ b\, \Delta x + c\, \Delta x^2 &= y_2 - y_1, \end{aligned}$$

from which we obtain

$$2c\, \Delta x^2 = y_0 - 2y_1 + y_2. \tag{8}$$

*Thomas Simpson (1710–1761), an English mathematics teacher whose name is wrongly attached to the rule that bears his name, was in his earlier years a professional astrologer and confidence man (one of his escapades forced him to flee to another town). His eventual success as a writer of elementary mathematics textbooks was greatly helped by accusations of plagiarism. This success enabled him to escape from poverty and leave his shady past behind him.

We now think of the parabola (5) as a close approximation to the curve $y = f(x)$ on the interval $[x_0, x_2]$, and we compute this part of the integral (1) accordingly,

$$\int_{x_0}^{x_2} f(x)\,dx \cong \int_{x_0}^{x_2} [a + b(x - x_1) + c(x - x_1)^2]\,dx$$

$$= \left[ax + \tfrac{1}{2}b(x - x_1)^2 + \tfrac{1}{3}c(x - x_1)^3\right]_{x_0}^{x_2}.$$

When this is evaluated in terms of Δx, we obtain

$$2a\,\Delta x + \tfrac{2}{3}c\,\Delta x^3.$$

By using (8) and the fact that $a = y_1$, we can write this in the form

$$2y_1\,\Delta x + \tfrac{1}{3}(y_0 - 2y_1 + y_2)\,\Delta x = \tfrac{1}{3}(y_0 + 4y_1 + y_2)\,\Delta x.$$

The same procedure can be applied to each of the intervals $[x_2, x_4]$, $[x_4, x_6]$, ... When the results are all added together, we get the approximation formula

$$\int_a^b f(x)\,dx \cong \tfrac{1}{3}(y_0 + 4y_1 + 2y_2 + \cdots + 4y_{n-1} + y_n)\,\Delta x,$$

which is called *Simpson's rule.* We specifically point out the structure of the expression in parentheses: y_0 and y_n occur with coefficient 1, the remaining y's with even subscripts occur with coefficient 2, and the y's with odd subscripts occur with coefficient 4.

Example 2 Use Simpson's rule with $n = 4$ to calculate an approximate value for the integral

$$\int_0^2 \frac{dx}{1 + x^4}.$$

This time we have $y = f(x) = 1/(1 + x^4)$ and $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, $x_4 = 2$. A simple table helps to keep the computations in order:

$y_0 = 1$	$y_0 = 1.000$
$y_1 = \frac{16}{17} = 0.941$	$4y_1 = 3.764$
$y_2 = \frac{1}{2} = 0.500$	$2y_2 = 1.000$
$y_3 = \frac{16}{97} = 0.165$	$4y_3 = 0.660$
$y_4 = \frac{1}{17} = 0.059$	$y_4 = \underline{0.059}$
	sum = 6.483

Simpson's rule now yields

$$\int_0^2 \frac{dx}{1 + x^4} \cong \tfrac{1}{6}(6.483) = 1.081.$$

Sometimes data is obtained from a scientific experiment with equally spaced observations. If this data represents isolated values of a function whose analytic expression is not known, then it may be wished to obtain an approximation to the integral of this function over the range of observation. Simpson's rule can be used in such a situation.

Example 3 If the experimental data is

x	0	0.5	1	1.5	2
y	1.0000	1.6487	2.7183	4.4817	7.3891

then

$$\int_0^2 y\,dx \cong \tfrac{1}{6}[1 + 4(1.6487) + 2(2.7183) + 4(4.4817) + 7.3891]$$

$$= 6.3912.$$

As a matter of fact, $y = e^x$ was the function used to generate this table of values, so the value of the integral is $e^2 - 1 = 6.3890560989$ to 10 decimal places.

Any serious study of a method of approximate calculation must include a detailed estimate of the magnitude of the error committed so that definite knowledge is available of the level of accuracy attained. We do not pursue this matter very far here, but merely state that the error in Simpson's rule is known to be at most

$$\frac{M(b-a)}{180}\,\Delta x^4, \tag{9}$$

where M is the maximum value of $f^{(4)}(x)$ on $[a, b]$. Derivations of this bound for the error can be found in books on numerical analysis. The power of Δx that appears in (9) tells us that if we reduce the width Δx by a factor of 10 (using 10 times as many subintervals), then we expect the maximum error to shrink by a factor of $10^4 = 10{,}000$. If we replace Δx in (9) by $(b - a)/n$, the bound (9) takes the form

$$\frac{M(b-a)^5}{180n^4}. \tag{10}$$

This formula enables us to impose a previously determined bound on the error by specifying a suitable value for n.

Example 3 *(continued)* We see that the actual error in the above calculation is about 0.0021 when $n = 4$. What value of n will guarantee that the error will be at most 0.0001?

In this case, assuming $f(x) = e^x$ really was the function underlying our data, then $f^{(4)}(x) = e^x$ and $M = e^2$. By (10) we therefore have

$$\frac{e^2 \cdot 2^5}{180n^4} = 0.0001,$$

so

$$n^4 = \frac{e^2 \cdot 2^5}{180}\,10^4 \qquad \text{or} \qquad n \cong 10.7.$$

Any integer $n \geq 11$ will therefore provide this level of accuracy.

Students who own calculators and enjoy working with them should note that the methods and problems of this section—and also of others to come, especially Section 14.5—provide plenty of raw material for these calculator enthusiasts.

PROBLEMS

1 Clearly,

$$\int_0^1 \sqrt{x}\,dx = \tfrac{2}{3} = 0.666\ldots.$$

Calculate the value of this integral approximately with $n = 4$ by using
(a) the trapezoidal rule (recall that $\sqrt{2} = 1.414\ldots$ and $\sqrt{3} = 1.732\ldots$);
(b) Simpson's rule.
Since the two rules are almost equally easy to apply, and Simpson's rule is usually more accurate, the trapezoidal rule is rarely used in practical computations.

2 Clearly,

$$\int_0^{\pi} \sin x\,dx = 2.$$

Calculate the value of this integral approximately by using Simpson's rule with $n = 4$.

3 The exact value of

$$\int_0^{\pi} \sqrt{\sin x}\,dx$$

is not known. Find its approximate value by using Simpson's rule with $n = 4$.

4 The exact value of

$$\int_1^5 \frac{e^x}{x}\,dx$$

is not known. Use Simpson's rule when $n = 4$ to find its approximate value.

5 The exact value of

$$\int_0^2 e^{-x^2}\,dx$$

is not known, but to 10 decimal places it is 0.8820813908. Calculate this integral approximately by using Simpson's rule with $n = 4$.

6 Find an approximate value for ln 2 by using the fact that

$$\ln 2 = \int_1^2 \frac{dx}{x}$$

and applying Simpson's rule with $n = 4$. (To 10 decimal places, $\ln 2 = 0.6931471806$.)

7 Use the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1 + x^2}$$

to find an approximate value for π by using Simpson's rule with $n = 4$. (To 10 decimal places, $\pi = 3.1415926536$.)

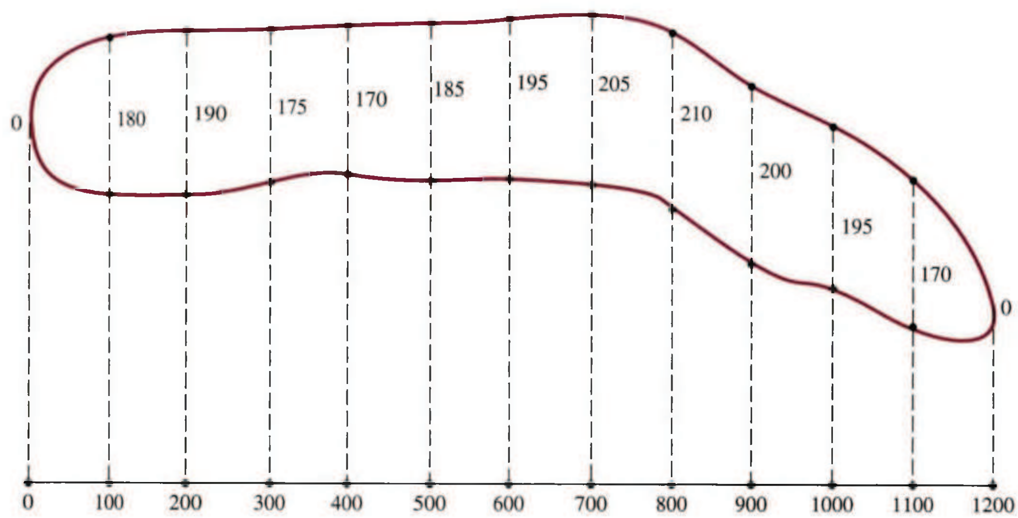

Figure 10.7 A dogleg fairway on a golf course.

8 In Example 3, what positive integers n will guarantee that the error is at most 0.000001?

9 The width, in feet, at equally spaced points along the fairway of a hole on a golf course is given in Fig. 10.7. The management wishes to estimate the number of square yards of the fairway as a basis for deciding how long it should take a groundskeeper to mow it. Use Simpson's rule to provide such an estimate.

10 Suppose that the three points on the curve in the derivation of Simpson's rule are collinear. Use (8) to show that in this case $c = 0$, and conclude that under this assumption the curve through the points is a straight line instead of a parabola.

11 Simpson's rule is designed to be exactly correct if $f(x)$ is a quadratic polynomial. It is a remarkable fact that it also gives an exact result for cubic polynomials. Prove this. Hint: Notice that it suffices to establish the statement for $n = 2$; then prove it in this case for the function $f(x) = x^3$; then extend it to any cubic polynomial.

12 Use formula (9) to prove the statement in Problem 11.

CHAPTER 10 REVIEW: FORMULAS, METHODS

Think through and learn the following.

1 The 15 basic formulas (write them down from memory).
2 Method of substitution.
3 Integrals of the form

$$\int \sin^m x \cos^n x\, dx, \qquad \int \tan^m x \sec^n x\, dx, \qquad \int \cot^m x \csc^n x\, dx.$$

4 The trigonometric substitutions $x = a \sin \theta$, $x = a \tan \theta$, $x = a \sec \theta$.
5 Completing the square: $(x + A)^2 = x^2 + 2Ax + A^2$.
6 Method of partial fractions.
7 Integration by parts.
8 Simpson's rule.

ADDITIONAL PROBLEMS FOR CHAPTER 10

SECTION 10.2

Find each of the following integrals.

1 $\int \sqrt{3x + 5}\, dx.$

2 $\int \frac{(\ln x)^6\, dx}{x}.$

3 $\int \frac{6x\, dx}{1 + 3x^2}.$

4 $\int \frac{e^{1/x}\, dx}{x^2}.$

5 $\int \cos(1 - 5x)\, dx.$

6 $\int \sin x \sin(\cos x)\, dx.$

7 $\int \frac{\sec \sqrt{x} \tan \sqrt{x}\, dx}{\sqrt{x}}.$

8 $\int \frac{x^3\, dx}{\sqrt{1 - x^8}}.$

9 $\int \frac{2x\, dx}{1 + x^4}.$

10 $\int \frac{x^2 + 5}{x^2 + 4}\, dx.$

11 $\int \cot 4x\, dx.$

12 $\int \frac{dx}{\sin 2x}.$

13 $\int \frac{dx}{x(\ln x)^2}.$

14 $\int \frac{dx}{3 - x}.$

15 $\int \frac{\sec^2 x\, dx}{\tan x}.$

16 $\int 10x^4 e^{x^5}\, dx.$

17 $\int \sin\left(\frac{3x - 5}{2}\right) dx.$

18 $\int \csc^2(2 - x)\, dx.$

19 $\int 6x^2 \cot x^3 \csc x^3\, dx.$

20 $\int \frac{\sec^2 x\, dx}{\sqrt{1 - \tan^2 x}}.$

21 $\int \frac{dx}{x[1 + (\ln x)^2]}.$

22 $\int \cot \pi x\, dx.$

23 $\int \frac{dx}{(3x + 5)^2}.$

24 $\int \tan x \sec^4 x\, dx.$

25 $\int \frac{dx}{3 - 2x}.$

26 $\int \frac{(e^x + 2x)\, dx}{e^x + x^2 - 2}.$

27 $\int x^2 \cos(1 + x^3)\, dx.$

28 $\int \sin(2 - x)\, dx.$

29 $\int x \csc^2(x^2 + 1)\, dx.$

30 $\int \frac{dx}{\sqrt{3 - 4x^2}}.$

31 $\int \frac{\cos x\, dx}{1 + \sin^2 x}.$

32 $\int \frac{dx}{1 + 4x^2}.$

33 $\int \frac{dx}{\tan 2x}.$

34 $\int (\csc x - 1)^2\, dx.$

35 $\int \frac{\tan^{-1} x\, dx}{1 + x^2}.$

36 $\int \sqrt[3]{3x - 2}\, dx.$

37 $\int \frac{dx}{2x + 1}.$

38 $\int \frac{(e^x - e^{-x})\, dx}{e^x + e^{-x}}.$

39 $\int e^{x/3}\,dx.$ **40** $\int \frac{dx}{\sec 2x}.$

41 $\int \frac{\sec^2 (\sin x)\,dx}{\sec x}.$

42 $\int (\csc x - \cot x)\csc x\,dx.$

43 $\int \frac{dx}{\sqrt{1-25x^2}}.$ **44** $\int \frac{dx}{16+25x^2}.$

45 $\int \frac{\sec x \tan x\,dx}{1+\sec^2 x}.$ **46** $\int (1+\sec x)^2\,dx.$

47 $\int \frac{(\ln x)^2\,dx}{x}.$ **48** $\int \frac{\cos x\,dx}{\sin^2 x}.$

49 $\int \frac{\sin x\,dx}{1+\cos x}.$ **50** $\int \frac{6\csc^2 x\,dx}{1-3\cot x}.$

51 $\int \frac{dx}{e^{3x}}.$ **52** $\int e^x \cos e^x\,dx.$

53 $\int \frac{\sin (\ln x)\,dx}{x}.$ **54** $\int \frac{\csc^2 \sqrt{x}\,dx}{\sqrt{x}}.$

55 $\int \frac{\csc 1/x \cot 1/x\,dx}{x^2}.$ **56** $\int \frac{4dx}{3+4x^2}.$

57 $\int \frac{e^{2x}\,dx}{1+e^{4x}}.$ **58** $\int \frac{x\,dx}{\sin x^2}.$

59 $\int x^3\sqrt{2+x^4}\,dx.$ **60** $\int \frac{x\,dx}{\sqrt{2-x^2}}.$

61 $\int \frac{(1+e^x)\,dx}{e^x+x}.$ **62** $\int xe^{x^2}\,dx.$

63 $\int \frac{2dx}{\sqrt{e^x}}.$ **64** $\int x\sin(1-x^2)\,dx.$

65 $\int \frac{dx}{\sin^2 x}.$ **66** $\int \frac{dx}{\sqrt{4-9x^2}}.$

67 $\int x\tan x^2\,dx.$ **68** $\int \frac{\sec^2 x\,dx}{\sqrt{\tan x}}.$

69 $\int \frac{x\,dx}{1+x^2}.$ **70** $\int 2e^{2x}\,dx.$

71 $\int xe^{3x^2-2}\,dx.$ **72** $\int 3x^2 \sin x^3\,dx.$

73 $\int \sec x(\sec x + \tan x)\,dx.$

74 $\int \frac{x^2\,dx}{9+x^6}.$

75 $\int x^{2/3}\sqrt{1+x^{5/3}}\,dx.$ **76** $\int \frac{4x^3\,dx}{1+x^4}.$

77 $\int \sec^2 x\, e^{\tan x}\,dx.$ **78** $\int x\sec^2 x^2\,dx.$

79 $\int (1+\cos x)^4 \sin x\,dx.$ **80** $\int \frac{(1+\cos x)\,dx}{x+\sin x}.$

81 $\int \cos(\tan x)\sec^2 x\,dx.$ **82** $\int \frac{\csc^2 (\ln x)\,dx}{x}.$

Compute each of the following definite integrals by making a suitable substitution and changing the limits of integration.

83 $\int_0^{\sqrt{2}/2} \frac{2x\,dx}{\sqrt{1-x^4}}.$ **84** $\int_0^{\sqrt{\pi}} x\sin x^2\,dx.$

85 $\int_{\pi/8}^{\pi/4} \cot 2x \csc^2 2x\,dx.$ **86** $\int_0^{\pi/2} \frac{\cos x\,dx}{1+\sin^2 x}.$

87 $\int_0^4 2x\sqrt{x^2+9}\,dx.$ **88** $\int_0^3 \frac{x\,dx}{\sqrt{x^2+16}}.$

SECTION 10.3

Calculate each of the following integrals.

89 $\int \sin^2 5x\,dx.$ **90** $\int \cos^4 3x\,dx.$

91 $\int \cos^2 7x\,dx.$ **92** $\int \sin^6 x\,dx.$

93 $\int \sin^5 x \cos^2 x\,dx.$ **94** $\int \sin^5 x\,dx.$

95 $\int \cos^3 4x\,dx.$ **96** $\int \cos^3 2x \sin 2x\,dx.$

97 $\int \frac{\cos^3 x\,dx}{\sin^4 x}.$ **98** $\int \frac{\sin^5 x\,dx}{\sqrt{\cos x}}.$

99 $\int \sin^{3/5} x \cos x\,dx.$ **100** $\int \sin^2 x \cos^4 x\,dx.$

101 $\int \sec^6 x\,dx.$ **102** $\int \frac{dx}{\cos^4 x}.$

103 $\int \tan^3 x \sec^7 x\,dx.$ **104** $\int \cot^4 x\,dx.$

105 $\int \cot^5 x\,dx.$ **106** $\int \frac{dx}{\sin^4 3x}.$

107 $\int (\sec 3x + \csc 3x)^2\,dx.$

108 $\int \frac{dx}{\sec x \tan x}.$

SECTION 10.4

Find each of the following integrals.

109 $\int \sqrt{3-x^2}\,dx.$ **110** $\int \frac{dx}{(a^2+x^2)^{3/2}}.$

111 $\int \frac{x^2\,dx}{a^2+x^2}.$ **112** $\int \frac{\sqrt{4-9x^2}}{x}\,dx.$

113 $\int x^3\sqrt{a^2-x^2}\,dx.$ **114** $\int \frac{x^3\,dx}{\sqrt{a^2+x^2}}.$

115 $\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx.$

116 $\int \frac{dx}{x^2\sqrt{a^2-x^2}}.$

117 $\int \frac{dx}{x^4\sqrt{a^2-x^2}}.$

118 $\int \frac{dx}{x^2+x^4}.$

119 $\int \frac{x^2\,dx}{(a^2+x^2)^2}.$

120 $\int x^3(a^2-x^2)^{3/2}\,dx.$

121 $\int \frac{dx}{x^2\sqrt{x^2-9}}.$

122 $\int \sqrt{x^2-1}\,dx.$

123 $\int \frac{dx}{(1-9x^2)^{3/2}}.$

124 $\int \frac{x^2\,dx}{\sqrt{a^2+x^2}}.$

125 $\int \frac{dx}{x\sqrt{9+4x^2}}.$

126 $\int \frac{dx}{\sqrt{9-(x-1)^2}}.$

127 $\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}}.$

128 $\int \frac{dx}{x^4\sqrt{a^2+x^2}}.$

129 $\int \frac{x^2\,dx}{(a^2+x^2)^{3/2}}.$

130 $\int \frac{\sqrt{a^2-x^2}}{x^4}\,dx.$

131 $\int \frac{x^2\,dx}{\sqrt{x^2-a^2}}.$

132 $\int \frac{x^3\,dx}{(x^2-a^2)^{3/2}}.$

SECTION 10.5

Calculate each of the following integrals.

133 $\int \frac{dx}{\sqrt{65-8x-x^2}}.$

134 $\int \frac{dx}{\sqrt{1+4x-x^2}}.$

135 $\int \frac{dx}{5x^2+10x+15}.$

136 $\int \frac{(3x-5)\,dx}{x^2+2x+2}.$

137 $\int \frac{dx}{\sqrt{2+2x-3x^2}}.$

138 $\int \frac{(1-x)\,dx}{\sqrt{8+2x-x^2}}.$

139 $\int \frac{x^2\,dx}{\sqrt{2x-x^2}}.$

140 $\int \frac{x\,dx}{\sqrt{x^2-4x+5}}.$

141 $\int \frac{dx}{3x^2-6x+15}.$

142 $\int \frac{(3x+4)\,dx}{\sqrt{2x+x^2}}.$

143 $\int \frac{dx}{(x-1)\sqrt{x^2-2x-3}}.$

144 $\int \frac{(2x-5)\,dx}{\sqrt{4x-x^2}}.$

145 $\int \frac{(3x+7)\,dx}{\sqrt{x^2+4x+8}}.$

146 $\int \sqrt{x^2+2x+2}\,dx.$

147 $\int \frac{(2x-3)\,dx}{(x^2+2x-3)^{3/2}}.$

148 $\int \sqrt{x^2-2x}\,dx.$

SECTION 10.6

Find each of the following integrals.

149 $\int \frac{16x+69}{x^2-x-12}\,dx.$

150 $\int \frac{3x-56}{x^2+3x-28}\,dx.$

151 $\int \frac{-8x-16}{4x^2-1}\,dx.$

152 $\int \frac{12x-63}{x^2-3x}\,dx.$

153 $\int \frac{3x^2-10x-60}{x^3+x^2-12x}\,dx.$

154 $\int \frac{8x^2+55x-25}{x^3-25x}\,dx.$

155 $\int \frac{-2x^2-18x+18}{x^3-9x}\,dx.$

156 $\int \frac{4x^2-2x-108}{x^3+5x^2-36x}\,dx.$

157 $\int \frac{-3x^3+x^2+2x+3}{x^4+x^3}\,dx.$

158 $\int \frac{9x^2-35x+28}{x^3-4x^2+4x}\,dx.$

159 $\int \frac{x^2-5x-8}{x^3+4x^2+8x}\,dx.$

160 $\int \frac{3x^2-5x+4}{x^3-x^2+x-1}\,dx.$

SECTION 10.7

Calculate the integrals in Problems 161–176 by the method of integration by parts.

161 $\int x^2\tan^{-1}x\,dx.$

162 $\int x^2\cos x\,dx.$

163 $\int \cos(\ln x)\,dx.$

164 $\int x\sin^2 x\,dx.$

165 $\int x^3\cos x\,dx.$

166 $\int \sqrt{1+x^2}\,dx.$

167 $\int \frac{\ln x\,dx}{(x+1)^2}.$

168 $\int \frac{xe^x\,dx}{(x+1)^2}.$

169 $\int \frac{x^3\,dx}{\sqrt{1+x^2}}.$

170 $\int x(x+3)^{10}\,dx.$

171 $\int e^{ax}\sin bx\,dx.$

172 $\int x^n\ln x\,dx \ (n\neq -1).$

173 $\int \frac{x\,dx}{e^x}.$

174 $\int x^2\sin x\,dx.$

175 $\int x^3e^{-2x}\,dx.$

176 $\int \ln(x+\sqrt{x^2+a^2})\,dx.$

177 Find the area under the curve $y = \sin\sqrt{x}$ from $x = 0$ to $x = \pi^2$.

178 Calculate the integral $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$ by using the identity

$$\frac{x^3}{\sqrt{1+x^2}} = \frac{x(1+x^2-1)}{\sqrt{1+x^2}} = x\sqrt{1+x^2} - \frac{x}{\sqrt{1+x^2}}.$$

Make sure your answer agrees with the result of Problem 169.

179 Calculate the integral $\int_0^a x^2\sqrt{a-x}\,dx$ (a) by using the substitution $u = \sqrt{a-x}$; (b) by parts.

180 Use integration by parts to show that

$$\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx.$$

Write $x^2 = -(-x^2) = -(a^2-x^2-a^2)$ in the numerator of the second integral and thereby obtain the formula

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}x\sqrt{a^2-x^2} + \tfrac{1}{2}a^2 \int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= \tfrac{1}{2}x\sqrt{a^2-x^2} + \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a} + c.$$

181 Use the method of Problem 180 to obtain the formula

$$\int (a^2-x^2)^n\,dx$$

$$= \frac{x(a^2-x^2)^n}{2n+1} + \frac{2a^2n}{2n+1}\int (a^2-x^2)^{n-1}\,dx.$$

***182** Use the idea of Problem 181 to obtain formula (9) in Section 10.6,

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{1}{2a^2(n-1)}\cdot\frac{x}{(a^2+x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}\int\frac{dx}{(a^2+x^2)^{n-1}}.$$

In the next three problems, derive the given reduction formula and apply it to the indicated special case.

183 (a) $\displaystyle\int x^m(\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx.$

(b) $\int x^5(\ln x)^3\,dx.$

184 (a) $\displaystyle\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx.$

(b) $\displaystyle\int x^3 e^{-2x}\,dx.$

185 (a) $\displaystyle\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx.$

(b) $\int \sec^3 x\,dx$ (see Problem 29 in Section 10.3).

APPENDIX 1: THE CATENARY, OR CURVE OF A HANGING CHAIN

As a specific example of the use of the methods of integration discussed in Section 10.4, we solve the classical problem of determining the exact shape of the curve assumed by a flexible chain (or cable, or rope) of uniform density which is suspended between two points and hangs under its own weight. This curve is called a *catenary,* from the Latin word for chain, *catena.**

Let the y-axis pass through the lowest point of the chain (Fig. 10.8), let s be the arc length from this point to a variable point (x, y), and let w_0 be the linear density (weight per unit length) of the chain. We obtain the differential equation of the catenary from the fact that the part of the chain between the lowest point and (x, y) is in static equilibrium under the action of three forces: the tension T_0 at the lowest point; the variable tension T at (x, y), which acts in the direction of the tangent because of the flexibility of the chain; and a downward force $w_0 s$ equal to the weight of the chain between these two points. Equating the horizontal component of T to T_0 and the vertical component of T to the weight of the chain gives

$$T\cos\theta = T_0 \quad\text{and}\quad T\sin\theta = w_0 s,$$

and by dividing we eliminate T and get $\tan\theta = w_0 s/T_0$ or

$$\frac{dy}{dx} = as, \quad\text{where}\quad a = \frac{w_0}{T_0}.$$

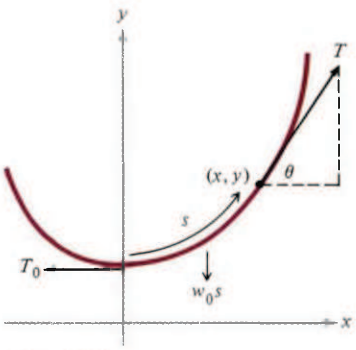

Figure 10.8

*The catenary problem is also solved in the optional Section 9.7 by using methods depending on hyperbolic functions. The solution given here does not depend on these methods and can therefore be understood by students who have omitted that optional section.

We next eliminate the variable s by differentiating with respect to x,

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{1}$$

This is the differential equation of the catenary.

We now solve equation (1) by two successive integrations. This process is facilitated by introducing the auxiliary variable $p = dy/dx$, so that (1) becomes

$$\frac{dp}{dx} = a\sqrt{1 + p^2}.$$

On separating variables and integrating, we get

$$\int \frac{dp}{\sqrt{1 + p^2}} = \int a\, dx. \tag{2}$$

To calculate the integral on the left, we make the trigonometric substitution $p = \tan \phi$, so that $dp = \sec^2 \phi\, d\phi$ and $\sqrt{1 + p^2} = \sec \phi$. Then

$$\int \frac{dp}{\sqrt{1 + p^2}} = \int \frac{\sec^2 \phi\, d\phi}{\sec \phi} = \int \sec \phi\, d\phi$$
$$= \ln(\sec \phi + \tan \phi) = \ln(\sqrt{1 + p^2} + p),$$

so (2) becomes

$$\ln(\sqrt{1 + p^2} + p) = ax + c_1.$$

Since $p = 0$ when $x = 0$, we see that $c_1 = 0$, so

$$\ln(\sqrt{1 + p^2} + p) = ax.$$

It is easy to solve this equation for p, which yields

$$\frac{dy}{dx} = p = \frac{1}{2}(e^{ax} - e^{-ax}),$$

and by integrating we obtain

$$y = \frac{1}{2a}(e^{ax} + e^{-ax}) + c_2.$$

If we now place the origin of the coordinate system in Fig. 10.8 at just the right level so that $y = 1/a$ when $x = 0$, then $c_2 = 0$ and our equation takes its final form,

$$y = \frac{1}{2a}(e^{ax} + e^{-ax}). \tag{3}$$

Equation (3) reveals the precise mathematical nature of the catenary and can be used as the basis for further investigations of its properties.*

The problem of finding the true shape of the catenary was proposed by James Bernoulli in 1690. Galileo had speculated long before that the curve was a parabola, but Huygens had shown in 1646 (at the age of 17), largely by physical reasoning, that this is not correct, without, however, shedding any light on what the shape might be. Bernoulli's challenge produced quick results, for in 1691 Leibniz, Huygens (now aged 62), and James's brother John all published independent solutions of the problem. John Bernoulli was ex-

*The hyperbolic cosine defined in Section 9.7 enables us to write the function (3) in the form

$$y = \frac{1}{a}\cosh ax.$$

ceedingly pleased that he had been successful in solving the problem, while his brother James, who proposed it, had failed. The taste of victory was still sweet 27 years later, as we see from this passage in a letter John wrote in 1718:

> The efforts of my brother were without success. For my part, I was more fortunate, for I found the skill (I say it without boasting; why should I conceal the truth?) to solve it in full. . . . It is true that it cost me study that robbed me of rest for an entire night. It was a great achievement for those days and for the slight age and experience I then had. The next morning, filled with joy, I ran to my brother, who was struggling miserably with this Gordian knot without getting anywhere, always thinking like Galileo that the catenary was a parabola. Stop! Stop! I say to him, don't torture yourself any more trying to prove the identity of the catenary with the parabola, since it is entirely false.

However, James evened the score by proving in the same year of 1691 that of all possible shapes a chain hanging between two fixed points might have, the catenary has the lowest center of gravity, and therefore the smallest potential energy. This was a very significant discovery, because it was the first hint of the profound idea that in some mysterious way the actual configurations of nature are those that minimize potential energy.

APPENDIX 2: WALLIS'S PRODUCT

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

As an application of integration by parts in Section 10.7, we obtained the following reduction formula:

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx. \tag{1}$$

This formula leads in an elementary but ingenious way to a very remarkable expression for the number $\pi/2$ as an infinite product,

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdots. \tag{2}$$

This expression was discovered by the English mathematician John Wallis in 1656 and is called *Wallis's product*. Apart from its intrinsic interest, formula (2) underlies other important developments in both pure and applied mathematics, so we prove it here.

If we define I_n by

$$I_n = \int_0^{\pi/2} \sin^n x\,dx,$$

then (1) tells us that

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{3}$$

It is clear that

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2} \qquad \text{and} \qquad I_1 = \int_0^{\pi/2} \sin x\,dx = 1.$$

We now distinguish the cases of even and odd subscripts, and use (3) to calculate I_{2n} and I_{2n+1}, as follows:

$$\begin{aligned} I_{2n} &= \frac{2n-1}{2n} I_{2n-2} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2} I_{2n-4} \\ &= \cdots = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{1}{2} I_0 \\ &= \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{\pi}{2}, \end{aligned} \tag{4}$$

and

$$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} I_{2n-3}$$

$$= \cdots = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{2}{3} I_1$$

$$= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1}. \tag{5}$$

As the next link in the chain of this reasoning, we need the fact that the ratio of these two quantities approaches 1 as $n \to \infty$,

$$\frac{I_{2n}}{I_{2n+1}} \to 1. \tag{6}$$

To establish this, we begin by noticing that on the interval $0 \le x \le \pi/2$ we have $0 \le \sin x \le 1$, and therefore

$$0 \le \sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x.$$

This implies that

$$0 < \int_0^{\pi/2} \sin^{2n+2} x \, dx \le \int_0^{\pi/2} \sin^{2n+1} x \, dx \le \int_0^{\pi/2} \sin^{2n} x \, dx,$$

or equivalently,

$$0 < I_{2n+2} \le I_{2n+1} \le I_{2n}. \tag{7}$$

If we divide through by I_{2n} and use the fact that by (3) we have

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2},$$

then (7) yields

$$\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1.$$

This implies that

$$\frac{I_{2n+1}}{I_{2n}} \to 1 \qquad \text{as} \qquad n \to \infty,$$

and this is equivalent to (6).

The final steps of the argument are as follows. On dividing (5) by (4), we obtain

$$\frac{I_{2n+1}}{I_{2n}} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{2}{\pi},$$

so

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \left(\frac{I_{2n}}{I_{2n+1}} \right).$$

On forming the limit as $n \to \infty$ and using (6), we obtain

$$\frac{\pi}{2} = \lim_{n \to \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1},$$

and this is what (2) means.

We also remark that Wallis's product (2) is equivalent to the formula

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\cdots=\frac{2}{\pi}. \tag{8}$$

This is easy to see if we write each number in parentheses on the left in factored form. This gives

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{6}\right)\left(1+\frac{1}{6}\right)\cdots=\frac{2}{\pi}$$

or

$$\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots=\frac{2}{\pi},$$

which is clearly equivalent to (2). Formula (8) will reappear in Appendix 1 at the end of Chapter 13 as a special case of another even more wonderful formula.*

APPENDIX 3: HOW LEIBNIZ DISCOVERED HIS FORMULA $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$

The area of the quarter-circle of radius 1 shown in Fig. 10.9 is obviously $\pi/4$. We follow Liebniz and calculate this area in a different way. The part that we actually calculate is the area A of the circular segment cut off by the chord OT, because the remainder of the quarter-circle is clearly an isosceles right triangle of area $\frac{1}{2}$.

We obtain the stated area A by integrating the sliverlike elements of area OPQ, where the arc PQ is considered to be so small that it is virtually straight. We think of OPQ as a triangle whose base is the segment PQ of length ds and whose height is the perpendicular distance OR from the vertex O to the base PQ extended. The two similar right triangles in the figure tell us that

$$\frac{ds}{dx}=\frac{OS}{OR} \qquad \text{or} \qquad OR\,ds = OS\,dx,$$

so the area dA of OPQ is

$$dA=\tfrac{1}{2}OR\,ds=\tfrac{1}{2}OS\,dx=\tfrac{1}{2}y\,dx,$$

*Wallis was Savilian Professor of Geometry at Oxford for 54 years, from 1649 until his death in 1703 at the age of 87, and played an important part in forming the climate of thought in which Newton flourished. He introduced negative and fractional exponents as well as the now-standard symbol ∞ for infinity, and was the first to treat conic sections as plane curves of the second degree. His infinite product stimulated his friend Lord Brouncker (first president of the Royal Society) to discover the astonishing formula

$$\frac{4}{\pi}=1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\cdots}}}}},$$

from which the theory of continued fractions later arose. [No one knows how Brouncker made this discovery, but a proof based on the work of Euler in the next century is given in the chapter on Brouncker in J. L. Coolidge's *The Mathematics of Great Amateurs* (Oxford University Press, 1949).] Among the activities of Wallis's later years was a lively quarrel with the famous philosopher Hobbes, who was under the impression that he had succeeded in squaring the circle and published his erroneous proof. Wallis promptly refuted it, but Hobbes was both arrogant and too ignorant to understand the refutation, and defended himself with a barrage of additional errors, as if a question about the validity of a mathematical proof could be settled by rhetoric and invective.

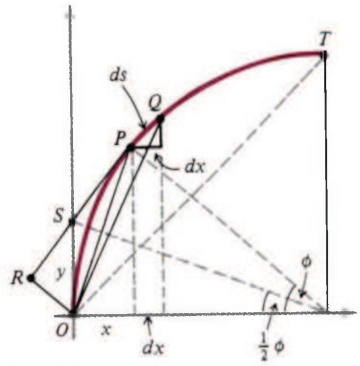

Figure 10.9

where y denotes the length of the segment OS. The element of area dA sweeps across the circular segment in question as x increases from 0 to 1, so

$$A = \int dA = \frac{1}{2}\int_0^1 y\,dx;$$

and integrating by parts in order to reverse the roles of x and y gives

$$A = \frac{1}{2}xy\Big]_0^1 - \frac{1}{2}\int_0^1 x\,dy = \frac{1}{2} - \frac{1}{2}\int_0^1 x\,dy, \tag{1}$$

where the limits on the two integrals are understood to be $y = 0$ and $y = 1$. To continue the calculation, we observe that since

$$y = \tan\tfrac{1}{2}\phi \qquad \text{and} \qquad x = 1 - \cos\phi = 2\sin^2\tfrac{1}{2}\phi,$$

the trigonometric identity

$$\tan^2\tfrac{1}{2}\phi = \frac{\sin^2\frac{1}{2}\phi}{\cos^2\frac{1}{2}\phi} = \sin^2\tfrac{1}{2}\phi\sec^2\tfrac{1}{2}\phi = \sin^2\tfrac{1}{2}\phi(1 + \tan^2\tfrac{1}{2}\phi)$$

yields

$$\frac{x}{2} = \frac{y^2}{1 + y^2}.$$

The version of the geometric series given in formula (13) in Section 9.5 enables us to write this as

$$\frac{x}{2} = y^2(1 - y^2 + y^4 - y^6 + \cdots) = y^2 - y^4 + y^6 - y^8 + \cdots,$$

so (1) becomes

$$\begin{aligned} A &= \frac{1}{2} - \int_0^1 (y^2 - y^4 + y^6 - y^8 + \cdots)\,dy \\ &= \frac{1}{2} - \left[\frac{1}{3}y^3 - \frac{1}{5}y^5 + \frac{1}{7}y^7 - \frac{1}{9}y^9 + \cdots\right]_0^1 \\ &= \frac{1}{2} - \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots\right) \\ &= \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots. \end{aligned}$$

When $\frac{1}{2}$ is added to this to account for the area of the isosceles right triangle, and the result is equated to the known area $\pi/4$ of the quarter-circle, we have Leibniz's formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Is it any wonder that he took great pleasure and pride in this discovery for the rest of his life?