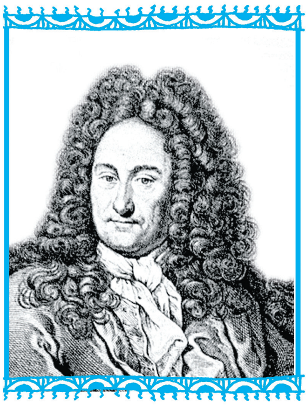
Chapter 7

INTEGRALS

❖ *Just as a mountaineer climbs a mountain – because it is there, so a good mathematics student studies new material because it is there.* — *JAMES B. BRISTOL* ❖

7.1 Introduction



G .W. Leibnitz
(1646 -1716)

Differential Calculus is centred on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.

If a function f is differentiable in an interval I, i.e., its derivative f' exists at each point of I, then a natural question arises that given f' at each point of I, can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function. Further, the formula that gives all these anti derivatives is called the ***indefinite integral*** of the function and such process of finding anti derivatives is called integration. Such type of problems arise in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types:

(a) the problem of finding a function whenever its derivative is given,

(b) the problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the ***Integral Calculus***.

There is a connection, known as the ***Fundamental Theorem of Calculus***, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering. The definite integral is also used to solve many interesting problems from various disciplines like economics, finance and probability.

In this Chapter, we shall confine ourselves to the study of indefinite and definite integrals and their elementary properties including some techniques of integration.

7.2 Integration as an Inverse Process of Differentiation

Integration is the inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called *integration* or *anti differentiation*.
Let us consider the following examples:

We know that $$\frac{d}{dx}(\sin x) = \cos x \qquad \text{... (1)}$$

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \qquad \text{... (2)}$$

and $$\frac{d}{dx}(e^x) = e^x \qquad \text{... (3)}$$

We observe that in (1), the function cos x is the derived function of sin x. We say that sin x is an anti derivative (or an integral) of cos x. Similarly, in (2) and (3), $\frac{x^3}{3}$ and e^x are the anti derivatives (or integrals) of x^2 and e^x, respectively. Again, we note that for any real number C, treated as constant function, its derivative is zero and hence, we can write (1), (2) and (3) as follows :

$$\frac{d}{dx}(\sin x + \text{C}) = \cos x\,,\ \frac{d}{dx}\left(\frac{x^3}{3} + \text{C}\right) = x^2 \text{ and } \frac{d}{dx}(e^x + \text{C}) = e^x$$

Thus, anti derivatives (or integrals) of the above cited functions are not unique. Actually, there exist infinitely many anti derivatives of each of these functions which can be obtained by choosing C arbitrarily from the set of real numbers. For this reason C is customarily referred to as ***arbitrary constant***. In fact, C is the ***parameter*** by varying which one gets different anti derivatives (or integrals) of the given function.

More generally, if there is a function F such that $\frac{d}{dx}\text{F}(x) = f(x)$, $\forall x \in$ I (interval), then for any arbitrary real number C, (also called *constant of integration*)

$$\frac{d}{dx}[\text{F}(x) + \text{C}] = f(x),\ x \in \text{I}$$

Thus, $\{F + C, C \in \mathbf{R}\}$ denotes a family of anti derivatives of f.

Remark Functions with same derivatives differ by a constant. To show this, let g and h be two functions having the same derivatives on an interval I.

Consider the function $f = g - h$ defined by $f(x) = g(x) - h(x)$, $\forall x \in I$

Then $\frac{df}{dx} = f' = g' - h'$ giving $f'(x) = g'(x) - h'(x)$ $\forall x \in I$

or $f'(x) = 0$, $\forall x \in I$ by hypothesis,

i.e., the rate of change of f with respect to x is zero on I and hence f is constant.

In view of the above remark, it is justified to infer that the family $\{F + C, C \in \mathbf{R}\}$ provides all possible anti derivatives of f.

We introduce a new symbol, namely, $\int f(x)\,dx$ which will represent the entire class of anti derivatives read as the indefinite integral of f with respect to x.

Symbolically, we write $\int f(x)\,dx = F(x) + C$.

Notation Given that $\frac{dy}{dx} = f(x)$, we write $y = \int f(x)\,dx$.

For the sake of convenience, we mention below the following symbols/terms/phrases with their meanings as given in the Table (7.1).

Table 7.1

Symbols/Terms/Phrases	Meaning
$\int f(x)\,dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)\,dx$	Integrand
x in $\int f(x)\,dx$	Variable of integration
Integrate	Find the integral
An integral of f	A function F such that $F'(x) = f(x)$
Integration	The process of finding the integral
Constant of Integration	Any real number C, considered as constant function

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down immediately the corresponding formulae (referred to as standard formulae) for the integrals of these functions, as listed below which will be used to find integrals of other functions.

Derivatives	**Integrals (Anti derivatives)**
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$;	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
Particularly, we note that	
$\frac{d}{dx}(x) = 1$;	$\int dx = x + C$
(ii) $\frac{d}{dx}(\sin x) = \cos x$;	$\int \cos x\, dx = \sin x + C$
(iii) $\frac{d}{dx}(-\cos x) = \sin x$;	$\int \sin x\, dx = -\cos x + C$
(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$;	$\int \sec^2 x\, dx = \tan x + C$
(v) $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$;	$\int \text{cosec}^2 x\, dx = -\cot x + C$
(vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$;	$\int \sec x \tan x\, dx = \sec x + C$
(vii) $\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x$;	$\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$
(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
(ix) $\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
(x) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
(xi) $\frac{d}{dx}(-\cot^{-1} x) = \frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xiii) $\frac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$; $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$

(xiv) $\frac{d}{dx}(e^x) = e^x$; $\int e^x dx = e^x + C$

(xv) $\frac{d}{dx}(\log|x|) = \frac{1}{x}$; $\int \frac{1}{x}\, dx = \log|x| + C$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$; $\int a^x dx = \frac{a^x}{\log a} + C$

Note In practice, we normally do not mention the interval over which the various functions are defined. However, in any specific problem one has to keep it in mind.

7.2.1 *Geometrical interpretation of indefinite integral*

Let $f(x) = 2x$. Then $\int f(x)\, dx = x^2 + C$. For different values of C, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + C$, where C is arbitrary constant, represents a family of integrals. By assigning different values to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

Clearly, for $C = 0$, we obtain $y = x^2$, a parabola with its vertex on the origin. The curve $y = x^2 + 1$ for $C = 1$ is obtained by shifting the parabola $y = x^2$ one unit along y-axis in positive direction. For $C = -1$, $y = x^2 - 1$ is obtained by shifting the parabola $y = x^2$ one unit along y-axis in the negative direction. Thus, for each positive value of C, each parabola of the family has its vertex on the positive side of the y-axis and for negative values of C, each has its vertex along the negative side of the y-axis. Some of these have been shown in the Fig 7.1.

Let us consider the intersection of all these parabolas by a line $x = a$. In the Fig 7.1, we have taken $a > 0$. The same is true when $a < 0$. If the line $x = a$ intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at P_0, P_1, P_2, P_{-1}, P_{-2} etc., respectively, then $\frac{dy}{dx}$ at these points equals $2a$. This indicates that the tangents to the curves at these points are parallel. Thus, $\int 2x\, dx = x^2 + C = F_C(x)$ (say), implies that

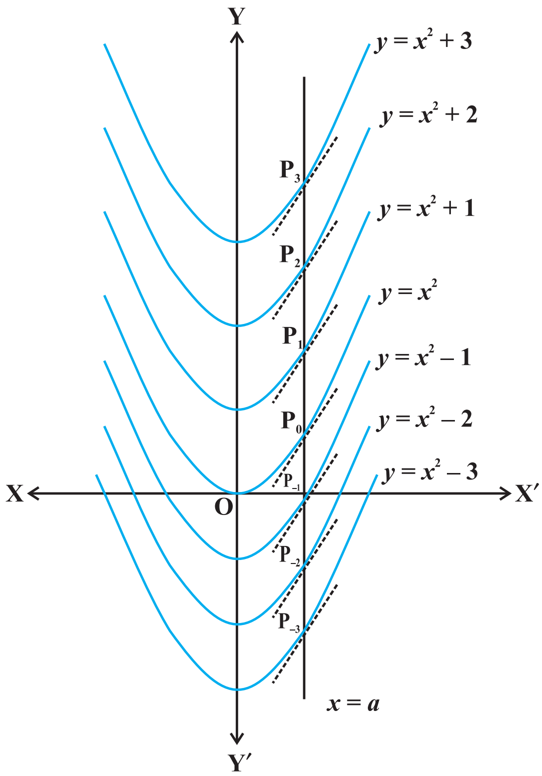


Fig 7.1

the tangents to all the curves $y = F_C(x)$, $C \in \mathbf{R}$, at the points of intersection of the curves by the line $x = a$, $(a \in \mathbf{R})$, are parallel.

Further, the following equation (statement) $\int f(x)\,dx = F(x) + C = y\ (\text{say})$, represents a family of curves. The different values of C will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

7.2.2 Some properties of indefinite integral

In this sub section, we shall derive some properties of indefinite integrals.

(I) The process of differentiation and integration are inverses of each other in the sense of the following results :

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

and $$\int f'(x)\,dx = f(x) + C,$$ where C is any arbitrary constant.

Proof Let F be any anti derivative of f, i.e.,

$$\frac{d}{dx}\mathrm{F}(x) = f(x)$$

Then $$\int f(x)\, dx = \mathrm{F}(x) + \mathrm{C}$$

Therefore $$\frac{d}{dx}\int f(x)\, dx = \frac{d}{dx}\left(\mathrm{F}(x) + \mathrm{C}\right)$$

$$= \frac{d}{dx}\mathrm{F}(x) = f(x)$$

Similarly, we note that

$$f'(x) = \frac{d}{dx} f(x)$$

and hence $$\int f'(x)\, dx = f(x) + \mathrm{C}$$

where C is arbitrary constant called constant of integration.

(II) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

Proof Let f and g be two functions such that

$$\frac{d}{dx}\int f(x)\, dx = \frac{d}{dx}\int g(x)\, dx$$

or $$\frac{d}{dx}\left[\int f(x)\, dx - \int g(x)\, dx\right] = 0$$

Hence $\int f(x)\, dx - \int g(x)\, dx = \mathrm{C}$, where C is any real number (Why?)

or $$\int f(x)\, dx = \int g(x)\, dx + \mathrm{C}$$

So the families of curves $\left\{\int f(x)\, dx + \mathrm{C}_1, \mathrm{C}_1 \in \mathbf{R}\right\}$

and $\left\{\int g(x)\, dx + \mathrm{C}_2, \mathrm{C}_2 \in \mathbf{R}\right\}$ are identical.

Hence, in this sense, $\int f(x)\, dx$ and $\int g(x)\, dx$ are equivalent.

Note The equivalence of the families $\left\{\int f(x)\,dx + C_1, C_1 \in \mathbf{R}\right\}$ and $\left\{\int g(x)\,dx + C_2, C_2 \in \mathbf{R}\right\}$ is customarily expressed by writing $\int f(x)\,dx = \int g(x)\,dx$, without mentioning the parameter.

(III) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Proof By Property (I), we have

$$\frac{d}{dx}\left[\int [f(x) + g(x)]\,dx\right] = f(x) + g(x) \qquad \dots (1)$$

On the otherhand, we find that

$$\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx$$

$$= f(x) + g(x) \qquad \dots (2)$$

Thus, in view of Property (II), it follows by (1) and (2) that

$$\int \left(f(x) + g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx.$$

(IV) For any real number k, $\int k\, f(x)\,dx = k\int f(x)\,dx$

Proof By the Property (I), $\frac{d}{dx}\int k\, f(x)\,dx = k\, f(x)$.

Also $\quad \frac{d}{dx}\left[k\int f(x)\,dx\right] = k\frac{d}{dx}\int f(x)\,dx = k\, f(x)$

Therefore, using the Property (II), we have $\int k\, f(x)\,dx = k\int f(x)\,dx$.

(V) Properties (III) and (IV) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int [k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x)]\,dx$$

$$= k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + ... + k_n\int f_n(x)\,dx.$$

To find an anti derivative of a given function, we search intuitively for a function whose derivative is the given function. The search for the requisite function for finding an anti derivative is known as integration by the method of inspection. We illustrate it through some examples.

Example 1 Write an anti derivative for each of the following functions using the method of inspection:

(i) $\cos 2x$ (ii) $3x^2 + 4x^3$ (iii) $\frac{1}{x}, x \neq 0$

Solution

(i) We look for a function whose derivative is cos 2*x*. Recall that

$$\frac{d}{dx} \sin 2x = 2 \cos 2x$$

or $$\cos 2x = \frac{1}{2}\frac{d}{dx}(\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

Therefore, an anti derivative of cos 2*x* is $\frac{1}{2}\sin 2x$.

(ii) We look for a function whose derivative is $3x^2 + 4x^3$. Note that

$$\frac{d}{dx}\left(x^3 + x^4\right) = 3x^2 + 4x^3.$$

Therefore, an anti derivative of $3x^2 + 4x^3$ is $x^3 + x^4$.

(iii) We know that

$$\frac{d}{dx}(\log x) = \frac{1}{x}, x > 0 \text{ and } \frac{d}{dx}[\log(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}, x < 0$$

Combining above, we get $\frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}, x \neq 0$

Therefore, $\int \frac{1}{x}\,dx = \log|x|$ is one of the anti derivatives of $\frac{1}{x}$.

Example 2 Find the following integrals:

(i) $\int \frac{x^3 - 1}{x^2}\,dx$ (ii) $\int (x^{\frac{2}{3}} + 1)\,dx$ (iii) $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})\,dx$

Solution

(i) We have

$$\int \frac{x^3 - 1}{x^2}\,dx = \int x\,dx - \int x^{-2}\,dx \qquad \text{(by Property V)}$$

$$= \left(\frac{x^{1+1}}{1+1} + C_1\right) - \left(\frac{x^{-2+1}}{-2+1} + C_2\right);\ C_1, C_2 \text{ are constants of integration}$$

$$= \frac{x^2}{2} + C_1 - \frac{x^{-1}}{-1} - C_2 = \frac{x^2}{2} + \frac{1}{x} + C_1 - C_2$$

$$= \frac{x^2}{2} + \frac{1}{x} + C, \text{ where } C = C_1 - C_2 \text{ is another constant of integration.}$$

Note From now onwards, we shall write only one constant of integration in the final answer.

(ii) We have

$$\int (x^{\frac{2}{3}} + 1)\, dx = \int x^{\frac{2}{3}}\, dx + \int dx$$

$$= \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + x + C = \frac{3}{5} x^{\frac{5}{3}} + x + C$$

(iii) We have $\int (x^{\frac{3}{2}} + 2\,e^x - \frac{1}{x})\, dx = \int x^{\frac{3}{2}}\, dx + \int 2\,e^x\, dx - \int \frac{1}{x} dx$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2\,e^x - \log|x| + C$$

$$= \frac{2}{5} x^{\frac{5}{2}} + 2\,e^x - \log|x| + C$$

Example 3 Find the following integrals:

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \operatorname{cosec} x\, (\operatorname{cosec} x + \cot x)\, dx$

(iii) $\int \frac{1 - \sin x}{\cos^2 x}\, dx$

Solution

(i) We have

$$\int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx$$

$$= -\cos x + \sin x + C$$

(ii) We have

$$\int (\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)\,dx = \int \operatorname{cosec}^2 x\,dx + \int \operatorname{cosec} x \cot x\,dx$$

$$= -\cot x - \operatorname{cosec} x + C$$

(iii) We have

$$\int \frac{1-\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x\,dx - \int \tan x \sec x\,dx$$

$$= \tan x - \sec x + C$$

Example 4 Find the anti derivative F of f defined by $f(x) = 4x^3 - 6$, where F (0) = 3

Solution One anti derivative of $f(x)$ is $x^4 - 6x$ since

$$\frac{d}{dx}(x^4 - 6x) = 4x^3 - 6$$

Therefore, the anti derivative F is given by

$$F(x) = x^4 - 6x + C, \text{ where C is constant.}$$

Given that $F(0) = 3$, which gives,

$$3 = 0 - 6 \times 0 + C \quad \text{or} \quad C = 3$$

Hence, the required anti derivative is the unique function F defined by

$$F(x) = x^4 - 6x + 3.$$

Remarks

(i) We see that if F is an anti derivative of f, then so is F + C, where C is any constant. Thus, if we know one anti derivative F of a function f, we can write down an infinite number of anti derivatives of f by adding any constant to F expressed by $F(x) + C, C \in \mathbf{R}$. In applications, it is often necessary to satisfy an additional condition which then determines a specific value of C giving unique anti derivative of the given function.

(ii) Sometimes, F is not expressible in terms of elementary functions viz., polynomial, logarithmic, exponential, trigonometric functions and their inverses etc. We are therefore blocked for finding $\int f(x)\,dx$. For example, it is not possible to find $\int e^{-x^2}\,dx$ by inspection since we can not find a function whose derivative is e^{-x^2}

(iii) When the variable of integration is denoted by a variable other than x, the integral formulae are modified accordingly. For instance

$$\int y^4\, dy = \frac{y^{4+1}}{4+1} + \mathrm{C} = \frac{1}{5} y^5 + \mathrm{C}$$

7.2.3 *Comparison between differentiation and integration*

1. Both are operations on functions.
2. Both satisfy the property of linearity, i.e.,

   (i) $\frac{d}{dx}[k_1 f_1(x) + k_2 f_2(x)] = k_1 \frac{d}{dx} f_1(x) + k_2 \frac{d}{dx} f_2(x)$

   (ii) $\int [k_1 f_1(x) + k_2 f_2(x)]\, dx = k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx$

   Here k_1 and k_2 are constants.
3. We have already seen that all functions are not differentiable. Similarly, all functions are not integrable. We will learn more about nondifferentiable functions and nonintegrable functions in higher classes.
4. The derivative of a function, when it exists, is a unique function. The integral of a function is not so. However, they are unique upto an additive constant, i.e., any two integrals of a function differ by a constant.
5. When a polynomial function P is differentiated, the result is a polynomial whose degree is 1 less than the degree of P. When a polynomial function P is integrated, the result is a polynomial whose degree is 1 more than that of P.
6. We can speak of the derivative at a point. We never speak of the integral at a point, we speak of the integral of a function over an interval on which the integral is defined as will be seen in Section 7.7.
7. The derivative of a function has a geometrical meaning, namely, the slope of the tangent to the corresponding curve at a point. Similarly, the indefinite integral of a function represents geometrically, a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines orthogonal (perpendicular) to the axis representing the variable of integration.
8. The derivative is used for finding some physical quantities like the velocity of a moving particle, when the distance traversed at any time t is known. Similarly, the integral is used in calculating the distance traversed when the velocity at time t is known.
9. Differentiation is a process involving limits. **So is integration,** as will be seen in Section 7.7.

10. The process of differentiation and integration are inverses of each other as discussed in Section 7.2.2 (i).

EXERCISE 7.1

Find an anti derivative (or integral) of the following functions by the method of inspection.

1. $\sin 2x$ **2.** $\cos 3x$ **3.** e^{2x}

4. $(ax + b)^2$ **5.** $\sin 2x - 4\ e^{3x}$

Find the following integrals in Exercises 6 to 20:

6. $\int (4\,e^{3x} + 1)\,dx$ **7.** $\int x^2 (1 - \frac{1}{x^2})\,dx$ **8.** $\int (ax^2 + bx + c)\,dx$

9. $\int (2x^2 + e^x)\,dx$ **10.** $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ **11.** $\int \frac{x^3 + 5x^2 - 4}{x^2}\,dx$

12. $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\,dx$ **13.** $\int \frac{x^3 - x^2 + x - 1}{x - 1}\,dx$ **14.** $\int (1 - x)\sqrt{x}\,dx$

15. $\int \sqrt{x}(3x^2 + 2x + 3)\,dx$ **16.** $\int (2x - 3\cos x + e^x)\,dx$

17. $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$ **18.** $\int \sec x\,(\sec x + \tan x)\,dx$

19. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\,dx$ **20.** $\int \frac{2 - 3\sin x}{\cos^2 x}\,dx.$

Choose the correct answer in Exercises 21 and 22.

21. The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

22. If $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

7.3 Methods of Integration

In previous section, we discussed integrals of those functions which were readily obtainable from derivatives of some functions. It was based on inspection, i.e., on the search of a function F whose derivative is f which led us to the integral of f. However, this method, which depends on inspection, is not very suitable for many functions. Hence, we need to develop additional techniques or methods for finding the integrals by reducing them into standard forms. Prominent among them are methods based on:

1. Integration by Substitution
2. Integration using Partial Fractions
3. Integration by Parts

7.3.1 *Integration by substitution*

In this section, we consider the method of integration by substitution.

The given integral $\int f(x)\,dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Consider $\qquad \mathrm{I} = \int f(x)\,dx$

Put $x = g(t)$ so that $\dfrac{dx}{dt} = g'(t)$.

We write $\qquad dx = g'(t)\,dt$

Thus $\qquad \mathrm{I} = \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated in the following examples.

Example 5 Integrate the following functions w.r.t. x:

(i) $\sin mx$ $\qquad$ (ii) $2x \sin (x^2 + 1)$

(iii) $\dfrac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$ $\qquad$ (iv) $\dfrac{\sin (\tan^{-1} x)}{1+x^2}$

Solution

(i) We know that derivative of mx is m. Thus, we make the substitution $mx = t$ so that $mdx = dt$.

Therefore, $\qquad \int \sin mx\,dx = \dfrac{1}{m}\int \sin t\,dt = -\dfrac{1}{m}\cos t + \mathrm{C} = -\dfrac{1}{m}\cos mx + \mathrm{C}$

(ii) Derivative of $x^2 + 1$ is $2x$. Thus, we use the substitution $x^2 + 1 = t$ so that $2x\, dx = dt$.

Therefore, $\int 2x \sin (x^2 + 1)\, dx = \int \sin t\, dt = -\cos t + C = -\cos (x^2 + 1) + C$

(iii) Derivative of $\sqrt{x}$ is $\frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Thus, we use the substitution

$\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}\, dx = dt$ giving $dx = 2t\, dt$.

Thus, $\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}\, dx = \int \frac{2t \tan^4 t \sec^2 t\, dt}{t} = 2 \int \tan^4 t \sec^2 t\, dt$

Again, we make another substitution $\tan t = u$ so that $\sec^2 t\, dt = du$

Therefore, $2 \int \tan^4 t \sec^2 t\, dt = 2 \int u^4\, du = 2 \frac{u^5}{5} + C$

$= \frac{2}{5} \tan^5 t + C$ (since $u = \tan t$)

$= \frac{2}{5} \tan^5 \sqrt{x} + C$ (since $t = \sqrt{x}$)

Hence, $\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}\, dx = \frac{2}{5} \tan^5 \sqrt{x} + C$

Alternatively, make the substitution $\tan \sqrt{x} = t$

(iv) Derivative of $\tan^{-1} x = \frac{1}{1+x^2}$. Thus, we use the substitution

$\tan^{-1} x = t$ so that $\frac{dx}{1+x^2} = dt$.

Therefore, $\int \frac{\sin (\tan^{-1} x)}{1+x^2}\, dx = \int \sin t\, dt = -\cos t + C = -\cos (\tan^{-1} x) + C$

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i) $\int \tan x\, dx = \log|\sec x| + C$

We have

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

Put $\cos x = t$ so that $\sin x\, dx = -dt$

Then $$\int \tan x\, dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$$

or $$\int \tan x\, dx = \log|\sec x| + C$$

(ii) $\displaystyle\int \cot x\, dx = \log|\sin x| + C$

We have $\displaystyle\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$

Put $\sin x = t$ so that $\cos x\, dx = dt$

Then $$\int \cot x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$$

(iii) $\displaystyle\int \sec x\, dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x\, dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

Put $\sec x + \tan x = t$ so that $\sec x\,(\tan x + \sec x)\, dx = dt$

Therefore, $\displaystyle\int \sec x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$

(iv) $\displaystyle\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

We have

$$\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)}\, dx$$

Put $\operatorname{cosec} x + \cot x = t$ so that $-\operatorname{cosec} x\,(\operatorname{cosec} x + \cot x)\, dx = dt$

So $$\int \operatorname{cosec} x\, dx = -\int \frac{dt}{t} = -\log|t| = -\log|\operatorname{cosec} x + \cot x| + C$$

$$= -\log\left|\frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec} x - \cot x}\right| + C$$

$$= \log|\operatorname{cosec} x - \cot x| + C$$

Example 6 Find the following integrals:

(i) $\displaystyle\int \sin^3 x \cos^2 x\, dx$ (ii) $\displaystyle\int \frac{\sin x}{\sin(x + a)}\, dx$ (iii) $\displaystyle\int \frac{1}{1 + \tan x}\, dx$

Solution

(i) We have

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \,(\sin x)\, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \,(\sin x)\, dx$$

Put $t = \cos x$ so that $dt = -\sin x \, dx$

Therefore, $\int \sin^2 x \cos^2 x \,(\sin x)\, dx = -\int (1 - t^2)\, t^2 \, dt$

$$= -\int (t^2 - t^4)\, dt = -\left(\frac{t^3}{3} - \frac{t^5}{5}\right) + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

(ii) Put $x + a = t$. Then $dx = dt$. Therefore

$$\int \frac{\sin x}{\sin (x + a)}\, dx = \int \frac{\sin (t - a)}{\sin t}\, dt$$

$$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t}\, dt$$

$$= \cos a \int dt - \sin a \int \cot t \, dt$$

$$= (\cos a)\, t - (\sin a)\left[\log\left|\sin t\right| + C_1\right]$$

$$= (\cos a)\,(x + a) - (\sin a)\left[\log\left|\sin (x + a)\right| + C_1\right]$$

$$= x \cos a + a \cos a - (\sin a) \log\left|\sin (x + a)\right| - C_1 \sin a$$

Hence, $\int \frac{\sin x}{\sin (x + a)}\, dx = x \cos a - \sin a \log |\sin (x + a)| + C,$

where, $C = -C_1 \sin a + a \cos a$, is another arbitrary constant.

(iii) $$\int \frac{dx}{1 + \tan x} = \int \frac{\cos x \, dx}{\cos x + \sin x}$$

$$= \frac{1}{2} \int \frac{(\cos x + \sin x + \cos x - \sin x)\, dx}{\cos x + \sin x}$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

$$= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx \qquad \text{... (1)}$$

Now, consider $I = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

Put $\cos x + \sin x = t$ so that $(\cos x - \sin x)\,dx = dt$

Therefore $\quad I = \int \frac{dt}{t} = \log|t| + C_2 = \log|\cos x + \sin x| + C_2$

Putting it in (1), we get

$$\int \frac{dx}{1+\tan x} = \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_1}{2} + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + C, \left(C = \frac{C_1}{2} + \frac{C_2}{2}\right)$$

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

1. $\frac{2x}{1+x^2}$ **2.** $\frac{(\log x)^2}{x}$ **3.** $\frac{1}{x + x\log x}$

4. $\sin x \sin(\cos x)$ **5.** $\sin(ax+b)\cos(ax+b)$

6. $\sqrt{ax+b}$ **7.** $x\sqrt{x+2}$ **8.** $x\sqrt{1+2x^2}$

9. $(4x+2)\sqrt{x^2+x+1}$ **10.** $\frac{1}{x-\sqrt{x}}$ **11.** $\frac{x}{\sqrt{x+4}}, x>0$

12. $(x^3-1)^{\frac{1}{3}}x^5$ **13.** $\frac{x^2}{(2+3x^3)^3}$ **14.** $\frac{1}{x(\log x)^m}, x>0, m \neq 1$

15. $\frac{x}{9-4x^2}$ **16.** e^{2x+3} **17.** $\frac{x}{e^{x^2}}$

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$ **19.** $\dfrac{e^{2x}-1}{e^{2x}+1}$ **20.** $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21. $\tan^2(2x-3)$ **22.** $\sec^2(7-4x)$ **23.** $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

24. $\dfrac{2\cos x-3\sin x}{6\cos x+4\sin x}$ **25.** $\dfrac{1}{\cos^2 x(1-\tan x)^2}$ **26.** $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

27. $\sqrt{\sin 2x}\cos 2x$ **28.** $\dfrac{\cos x}{\sqrt{1+\sin x}}$ **29.** $\cot x \log \sin x$

30. $\dfrac{\sin x}{1+\cos x}$ **31.** $\dfrac{\sin x}{(1+\cos x)^2}$ **32.** $\dfrac{1}{1+\cot x}$

33. $\dfrac{1}{1-\tan x}$ **34.** $\dfrac{\sqrt{\tan x}}{\sin x\cos x}$ **35.** $\dfrac{(1+\log x)^2}{x}$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$ **37.** $\dfrac{x^3\sin(\tan^{-1}x^4)}{1+x^8}$

Choose the correct answer in Exercises 38 and 39.

38. $\displaystyle\int \frac{10x^9+10^x\log_{e^{10}} dx}{x^{10}+10^x}$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$ (D) $\log(10^x + x^{10}) + C$

39. $\displaystyle\int \frac{dx}{\sin^2 x\cos^2 x}$ equals

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

7.3.2 *Integration using trigonometric identities*

When the integrand involves some trigonometric functions, we use some known identities to find the integral as illustrated through the following example.

Example 7 Find (i) $\int \cos^2 x\,dx$ (ii) $\int \sin 2x \cos 3x\,dx$ (iii) $\int \sin^3 x\,dx$

Solution

(i) Recall the identity $\cos 2x = 2\cos^2 x - 1$, which gives

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Therefore, $$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

(ii) Recall the identity $\sin x \cos y = \frac{1}{2}[\sin (x+y) + \sin (x-y)]$ (Why?)

Then $$\int \sin 2x \cos 3x\,dx = \frac{1}{2}\left[\int \sin 5x\,dx \bullet \int \sin x\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right] + C$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$$

(iii) From the identity $\sin 3x = 3\sin x - 4\sin^3 x$, we find that

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Therefore, $$\int \sin^3 x\,dx = \frac{3}{4}\int \sin x\,dx - \frac{1}{4}\int \sin 3x\,dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

Alternatively, $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx$

Put $\cos x = t$ so that $-\sin x\,dx = dt$

Therefore, $$\int \sin^3 x\,dx = -\int \left(1-t^2\right) dt = -\int dt + \int t^2\,dt = -t + \frac{t^3}{3} + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

Remark It can be shown using trigonometric identities that both answers are equivalent.

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

1. $\sin^2 (2x + 5)$
2. $\sin 3x \cos 4x$
3. $\cos 2x \cos 4x \cos 6x$
4. $\sin^3 (2x + 1)$
5. $\sin^3 x \cos^3 x$
6. $\sin x \sin 2x \sin 3x$
7. $\sin 4x \sin 8x$
8. $\dfrac{1 - \cos x}{1 + \cos x}$
9. $\dfrac{\cos x}{1 + \cos x}$
10. $\sin^4 x$
11. $\cos^4 2x$
12. $\dfrac{\sin^2 x}{1 + \cos x}$
13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$
14. $\dfrac{\cos x - \sin x}{1 + \sin 2x}$
15. $\tan^3 2x \sec 2x$
16. $\tan^4 x$
17. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$
18. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$
19. $\dfrac{1}{\sin x \cos^3 x}$
20. $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$
21. $\sin^{-1} (\cos x)$
22. $\dfrac{1}{\cos (x - a) \cos (x - b)}$

Choose the correct answer in Exercises 23 and 24.

23. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to

(A) $\tan x + \cot x + C$ (B) $\tan x + \operatorname{cosec} x + C$
(C) $- \tan x + \cot x + C$ (D) $\tan x + \sec x + C$

24. $\displaystyle\int \frac{e^x (1 + x)}{\cos^2 (e^x x)} dx$ equals

(A) $- \cot (ex^x) + C$ (B) $\tan (xe^x) + C$
(C) $\tan (e^x) + C$ (D) $\cot (e^x) + C$

7.4 Integrals of Some Particular Functions

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

(1) $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + C$

(2) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$

(3) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

(4) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$

(5) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

(6) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$

We now prove the above results:

(1) We have $\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)}$

$$= \frac{1}{2a} \left[\frac{(x+a) - (x-a)}{(x-a)(x+a)} \right] = \frac{1}{2a} \left[\frac{1}{x-a} - \frac{1}{x+a} \right]$$

Therefore, $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right]$

$$= \frac{1}{2a} \left[\log |(x-a)| - \log |(x+a)| \right] + C$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(2) In view of (1) above, we have

$$\frac{1}{a^2 - x^2} = \frac{1}{2a} \left[\frac{(a+x) + (a-x)}{(a+x)(a-x)} \right] = \frac{1}{2a} \left[\frac{1}{a-x} + \frac{1}{a+x} \right]$$

Therefore, $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\left[\int \frac{dx}{a-x} + \int \frac{dx}{a+x}\right]$

$$= \frac{1}{2a}[-\log|a-x| + \log|a+x|] + C$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

Note The technique used in (1) will be explained in Section 7.5.

(3) Put $x = a \tan\theta$. Then $dx = a \sec^2\theta \, d\theta$.

Therefore, $\int \frac{dx}{x^2 + a^2} = \int \frac{a \sec^2\theta \, d\theta}{a^2 \tan^2\theta + a^2}$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4) Let $x = a \sec\theta$. Then $dx = a \sec\theta \tan\theta \, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec\theta \tan\theta \, d\theta}{\sqrt{a^2 \sec^2\theta - a^2}}$

$$= \int \sec\theta \, d\theta = \log|\sec\theta + \tan\theta| + C_1$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| - \log|a| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C, \text{ where } C = C_1 - \log|a|$$

(5) Let $x = a \sin\theta$. Then $dx = a \cos\theta \, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos\theta \, d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}}$

$$= \int d\theta = \theta + C = \sin^{-1}\frac{x}{a} + C$$

(6) Let $x = a \tan\theta$. Then $dx = a \sec^2\theta \, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \sec^2\theta \, d\theta}{\sqrt{a^2 \tan^2\theta + a^2}}$

$$= \int \sec\theta \, d\theta = \log|(\sec\theta + \tan\theta)| + C_1$$