$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right|+C_1$$

$$= \log\left|x+\sqrt{x^2+a^2}\right| - \log|a| + C_1$$

$$= \log\left|x+\sqrt{x^2+a^2}\right| + C \text{, where } C = C_1 - \log|a|$$

Applying these standard formulae, we now obtain some more formulae which are useful from applications point of view and can be applied directly to evaluate other integrals.

(7) To find the integral $\int \frac{dx}{ax^2+bx+c}$, we write

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

Now, put $x+\frac{b}{2a} = t$ so that $dx = dt$ and writing $\frac{c}{a}-\frac{b^2}{4a^2} = \pm k^2$. We find the integral reduced to the form $\frac{1}{a}\int\frac{dt}{t^2 \pm k^2}$ depending upon the sign of $\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)$ and hence can be evaluated.

(8) To find the integral of the type $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, proceeding as in (7), we obtain the integral using the standard formulae.

(9) To find the integral of the type $\int \frac{px+q}{ax^2+bx+c}dx$, where p, q, a, b, c are constants, we are to find real numbers A, B such that

$$px+q = A\frac{d}{dx}(ax^2+bx+c) + B = A\,(2ax+b) + B$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

(10) For the evaluation of the integral of the type $\int \frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}}$, we proceed as in (9) and transform the integral into known standard forms.

Let us illustrate the above methods by some examples.

Example 8 Find the following integrals:

(i) $\int \frac{dx}{x^2-16}$ (ii) $\int \frac{dx}{\sqrt{2x-x^2}}$

Solution

(i) We have $\int \frac{dx}{x^2-16} = \int \frac{dx}{x^2-4^2} = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + C$ [by 7.4 (1)]

(ii) $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

Put $x - 1 = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + C$ [by 7.4 (5)]

$$= \sin^{-1}(x-1) + C$$

Example 9 Find the following integrals :

(i) $\int \frac{dx}{x^2-6x+13}$ (ii) $\int \frac{dx}{3x^2+13x-10}$ (iii) $\int \frac{dx}{\sqrt{5x^2-2x}}$

Solution

(i) We have $x^2 - 6x + 13 = x^2 - 6x + 3^2 - 3^2 + 13 = (x-3)^2 + 4$

So, $\int \frac{dx}{x^2-6x+13} = \int \frac{1}{(x-3)^2+2^2}\,dx$

Let $x - 3 = t$. Then $dx = dt$

Therefore, $\int \frac{dx}{x^2-6x+13} = \int \frac{dt}{t^2+2^2} = \frac{1}{2}\tan^{-1}\frac{t}{2} + C$ [by 7.4 (3)]

$$= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

(ii) The given integral is of the form 7.4 (7). We write the denominator of the integrand,

$$3x^2 + 13x - 10 = 3\left(x^2 + \frac{13x}{3} - \frac{10}{3}\right)$$

$$= 3\left[\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2\right] \text{ (completing the square)}$$

Thus $\displaystyle\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3}\int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}$

Put $x + \frac{13}{6} = t$. Then $dx = dt$.

Therefore, $\displaystyle\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3}\int \frac{dt}{t^2 - \left(\frac{17}{6}\right)^2}$

$$= \frac{1}{3 \times 2 \times \frac{17}{6}} \log\left|\frac{t - \frac{17}{6}}{t + \frac{17}{6}}\right| + C_1 \qquad \text{[by 7.4 (i)]}$$

$$= \frac{1}{17} \log\left|\frac{x + \frac{13}{6} - \frac{17}{6}}{x + \frac{13}{6} + \frac{17}{6}}\right| + C_1$$

$$= \frac{1}{17} \log\left|\frac{6x - 4}{6x + 30}\right| + C_1$$

$$= \frac{1}{17} \log\left|\frac{3x - 2}{x + 5}\right| + C_1 + \frac{1}{17}\log\frac{1}{3}$$

$$= \frac{1}{17} \log\left|\frac{3x - 2}{x + 5}\right| + C, \text{ where } C = C_1 + \frac{1}{17}\log\frac{1}{3}$$

(iii) We have $\int \frac{dx}{\sqrt{5x^2-2x}} = \int \frac{dx}{\sqrt{5\left(x^2-\frac{2x}{5}\right)}}$

$$= \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{\left(x-\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2}} \quad \text{(completing the square)}$$

Put $x-\frac{1}{5}=t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{5x^2-2x}} = \frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{t^2-\left(\frac{1}{5}\right)^2}}$

$$= \frac{1}{\sqrt{5}}\log\left|t+\sqrt{t^2-\left(\frac{1}{5}\right)^2}\right|+C \quad \text{[by 7.4 (4)]}$$

$$= \frac{1}{\sqrt{5}}\log\left|x-\frac{1}{5}+\sqrt{x^2-\frac{2x}{5}}\right|+C$$

Example 10 Find the following integrals:

(i) $\int \frac{x+2}{2x^2+6x+5}\,dx$ (ii) $\int \frac{x+3}{\sqrt{5-4x+x^2}}\,dx$

Solution

(i) Using the formula 7.4 (9), we express

$$x+2 = A\frac{d}{dx}\left(2x^2+6x+5\right)+B = A(4x+6)+B$$

Equating the coefficients of x and the constant terms from both sides, we get

$4A = 1$ and $6A + B = 2$ or $A = \frac{1}{4}$ and $B = \frac{1}{2}$.

Therefore, $\int \frac{x+2}{2x^2+6x+5} = \frac{1}{4}\int \frac{4x+6}{2x^2+6x+5}dx + \frac{1}{2}\int \frac{dx}{2x^2+6x+5}$

$$= \frac{1}{4}I_1 + \frac{1}{2}I_2 \quad \text{(say)} \quad \ldots (1)$$

In I_1, put $2x^2 + 6x + 5 = t$, so that $(4x + 6)\, dx = dt$

Therefore, $$I_1 = \int \frac{dt}{t} = \log|t| + C_1$$

$$= \log|2x^2 + 6x + 5| + C_1 \quad \text{... (2)}$$

and $$I_2 = \int \frac{dx}{2x^2 + 6x + 5} = \frac{1}{2}\int \frac{dx}{x^2 + 3x + \frac{5}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

Put $x + \frac{3}{2} = t$, so that $dx = dt$, we get

$$I_2 = \frac{1}{2}\int \frac{dt}{t^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2 \times \frac{1}{2}} \tan^{-1} 2t + C_2 \quad \text{[by 7.4 (3)]}$$

$$= \tan^{-1} 2\left(x + \frac{3}{2}\right) + C_2 = \tan^{-1}(2x + 3) + C_2 \quad \text{... (3)}$$

Using (2) and (3) in (1), we get

$$\int \frac{x + 2}{2x^2 + 6x + 5}\, dx = \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{2}\tan^{-1}(2x + 3) + C$$

where, $$C = \frac{C_1}{4} + \frac{C_2}{2}$$

(ii) This integral is of the form given in 7.4 (10). Let us express

$$x + 3 = A\frac{d}{dx}(5 - 4x - x^2) + B = A(-4 - 2x) + B$$

Equating the coefficients of x and the constant terms from both sides, we get

$-2A = 1$ and $-4A + B = 3$, i.e., $A = -\frac{1}{2}$ and $B = 1$

Therefore, $\int \frac{x+3}{\sqrt{5-4x-x^2}}\,dx = -\frac{1}{2}\int \frac{(-4-2x)\,dx}{\sqrt{5-4x-x^2}} + \int \frac{dx}{\sqrt{5-4x-x^2}}$

$$= -\frac{1}{2}\,\mathrm{I}_1 + \mathrm{I}_2 \qquad \ldots (1)$$

In I_1, put $5 - 4x - x^2 = t$, so that $(-4 - 2x)\,dx = dt$.

Therefore, $\mathrm{I}_1 = \int \frac{(-4-2x)\,dx}{\sqrt{5-4x-x^2}} = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + \mathrm{C}_1$

$$= 2\sqrt{5-4x-x^2} + \mathrm{C}_1 \qquad \ldots (2)$$

Now consider $\mathrm{I}_2 = \int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}}$

Put $x + 2 = t$, so that $dx = dt$.

Therefore, $\mathrm{I}_2 = \int \frac{dt}{\sqrt{3^2-t^2}} = \sin^{-1}\frac{t}{3} + \mathrm{C}_2$ [by 7.4 (5)]

$$= \sin^{-1}\frac{x+2}{3} + \mathrm{C}_2 \qquad \ldots (3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{\sqrt{5-4x-x^2}} = -\sqrt{5-4x-x^2} + \sin^{-1}\frac{x+2}{3} + \mathrm{C}, \text{ where } \mathrm{C} = \mathrm{C}_2 - \frac{\mathrm{C}_1}{2}$$

EXERCISE 7.4

Integrate the functions in Exercises 1 to 23.

1. $\frac{3x^2}{x^6+1}$ **2.** $\frac{1}{\sqrt{1+4x^2}}$ **3.** $\frac{1}{\sqrt{(2-x)^2+1}}$

4. $\frac{1}{\sqrt{9-25x^2}}$ **5.** $\frac{3x}{1+2x^4}$ **6.** $\frac{x^2}{1-x^6}$

7. $\frac{x-1}{\sqrt{x^2-1}}$ **8.** $\frac{x^2}{\sqrt{x^6+a^6}}$ **9.** $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

10. $\dfrac{1}{\sqrt{x^2+2x+2}}$ **11.** $\dfrac{1}{9x^2+6x+5}$ **12.** $\dfrac{1}{\sqrt{7-6x-x^2}}$

13. $\dfrac{1}{\sqrt{(x-1)(x-2)}}$ **14.** $\dfrac{1}{\sqrt{8+3x-x^2}}$ **15.** $\dfrac{1}{\sqrt{(x-a)(x-b)}}$

16. $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$ **17.** $\dfrac{x+2}{\sqrt{x^2-1}}$ **18.** $\dfrac{5x-2}{1+2x+3x^2}$

19. $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$ **20.** $\dfrac{x+2}{\sqrt{4x-x^2}}$ **21.** $\dfrac{x+2}{\sqrt{x^2+2x+3}}$

22. $\dfrac{x+3}{x^2-2x-5}$ **23.** $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$.

Choose the correct answer in Exercises 24 and 25.

24. $\displaystyle\int \frac{dx}{x^2+2x+2}$ equals

(A) $x\tan^{-1}(x+1)+C$ (B) $\tan^{-1}(x+1)+C$

(C) $(x+1)\tan^{-1}x+C$ (D) $\tan^{-1}x+C$

25. $\displaystyle\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\dfrac{1}{9}\sin^{-1}\left(\dfrac{9x-8}{8}\right)+C$ (B) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{8x-9}{9}\right)+C$

(C) $\dfrac{1}{3}\sin^{-1}\left(\dfrac{9x-8}{8}\right)+C$ (D) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{9x-8}{9}\right)+C$

7.5 Integration by Partial Fractions

Recall that a rational function is defined as the ratio of two polynomials in the form $\dfrac{P(x)}{Q(x)}$, where P (x) and Q(x) are polynomials in x and Q$(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational functions can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$,

where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function. As we know how to integrate polynomials, the integration of any rational function is reduced to the integration of a proper rational function. The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors. Assume that we want to evaluate $\int \frac{P(x)}{Q(x)} dx$, where $\frac{P(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods. The following Table 7.2 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.2

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where $x^2 + bx + c$ cannot be factorised further	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$,

In the above table, A, B and C are real numbers to be determined suitably.

Example 11 Find $\int \frac{dx}{(x+1)(x+2)}$

Solution The integrand is a proper rational function. Therefore, by using the form of partial fraction [Table 7.2 (i)], we write

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad \text{... (1)}$$

where, real numbers A and B are to be determined suitably. This gives

$$1 = A\,(x+2) + B\,(x+1).$$

Equating the coefficients of x and the constant term, we get

$$A + B = 0$$

and $$2A + B = 1$$

Solving these equations, we get $A = 1$ and $B = -1$.

Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$$

Therefore, $$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log|x+1| - \log|x+2| + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C$$

Remark The equation (1) above is an identity, i.e. a statement true for all (permissible) values of x. Some authors use the symbol '$\equiv$' to indicate that the statement is an identity and use the symbol '=' to indicate that the statement is an equation, i.e., to indicate that the statement is true only for certain values of x.

Example 12 Find $\int \frac{x^2+1}{x^2-5x+6}\,dx$

Solution Here the integrand $\frac{x^2+1}{x^2-5x+6}$ is not proper rational function, so we divide $x^2 + 1$ by $x^2 - 5x + 6$ and find that

$$\frac{x^2+1}{x^2-5x+6} = 1+\frac{5x-5}{x^2-5x+6} = 1+\frac{5x-5}{(x-2)(x-3)}$$

Let $$\frac{5x-5}{(x-2)(x-3)} = \frac{A}{x-2}+\frac{B}{x-3}$$

So that $$5x-5 = A(x-3)+B(x-2)$$

Equating the coefficients of x and constant terms on both sides, we get A + B = 5 and 3A + 2B = 5. Solving these equations, we get A = – 5 and B = 10

Thus, $$\frac{x^2+1}{x^2-5x+6} = 1-\frac{5}{x-2}+\frac{10}{x-3}$$

Therefore, $$\int \frac{x^2+1}{x^2-5x+6}\,dx = \int dx - 5\int\frac{1}{x-2}\,dx + 10\int\frac{dx}{x-3}$$

$$= x - 5\log|x-2| + 10\log|x-3| + C.$$

Example 13 Find $\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$

Solution The integrand is of the type as given in Table 7.2 (4). We write

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$$

So that $$3x-2 = A(x+1)(x+3)+B(x+3)+C(x+1)^2$$

$$= A(x^2+4x+3)+B(x+3)+C(x^2+2x+1)$$

Comparing coefficient of x^2, x and constant term on both sides, we get A + C = 0, 4A + B + 2C = 3 and 3A + 3B + C = – 2. Solving these equations, we get

$A=\frac{11}{4}, B=\frac{-5}{2}$ and $C=\frac{-11}{4}$. Thus the integrand is given by

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4(x+1)}-\frac{5}{2(x+1)^2}-\frac{11}{4(x+3)}$$

Therefore, $$\int\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4}\int\frac{dx}{x+1}-\frac{5}{2}\int\frac{dx}{(x+1)^2}-\frac{11}{4}\int\frac{dx}{x+3}$$

$$= \frac{11}{4}\log|x+1|+\frac{5}{2(x+1)}-\frac{11}{4}\log|x+3|+C$$

$$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2(x+1)}+C$$

Example 14 Find $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

Solution Consider $\frac{x^2}{(x^2+1)(x^2+4)}$ and put $x^2 = y$.

Then $$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$$

Write $$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$$

So that $$y = A\,(y+4) + B\,(y+1)$$

Comparing coefficients of y and constant terms on both sides, we get A + B = 1 and 4A + B = 0, which give

$$A = -\frac{1}{3} \quad \text{and} \quad B = \frac{4}{3}$$

Thus, $$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

Therefore, $$\int \frac{x^2 dx}{(x^2+1)(x^2+4)} = -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{4}{3}\times\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C$$

In the above example, the substitution was made only for the partial fraction part and not for the integration part. Now, we consider an example, where the integration involves a combination of the substitution method and the partial fraction method.

Example 15 Find $\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi} d\phi$

Solution Let $y = \sin\phi$

Then $$dy = \cos\phi \; d\phi$$

Therefore, $$\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi}\, d\phi = \int \frac{(3y-2)\, dy}{5 - (1 - y^2) - 4y}$$

$$= \int \frac{3y-2}{y^2 - 4y + 4}\, dy$$

$$= \int \frac{3y-2}{(y-2)^2} = \text{I (say)}$$

Now, we write $$\frac{3y-2}{(y-2)^2} = \frac{A}{y-2} + \frac{B}{(y-2)^2} \qquad \text{[by Table 7.2 (2)]}$$

Therefore, $$3y - 2 = A\,(y-2) + B$$

Comparing the coefficients of y and constant term, we get $A = 3$ and $B - 2A = -2$, which gives $A = 3$ and $B = 4$.

Therefore, the required integral is given by

$$I = \int \left[\frac{3}{y-2} + \frac{4}{(y-2)^2}\right] dy = 3\int \frac{dy}{y-2} + 4\int \frac{dy}{(y-2)^2}$$

$$= 3\log|y-2| + 4\left(-\frac{1}{y-2}\right) + C$$

$$= 3\log|\sin\phi - 2| + \frac{4}{2 - \sin\phi} + C$$

$$= 3\log(2 - \sin\phi) + \frac{4}{2 - \sin\phi} + C \quad \text{(since, } 2 - \sin\phi \text{ is always positive)}$$

Example 16 Find $\int \frac{x^2 + x + 1\, dx}{(x+2)(x^2+1)}$

Solution The integrand is a proper rational function. Decompose the rational function into partial fraction [Table 2.2(5)]. Write

$$\frac{x^2 + x + 1}{(x^2+1)(x+2)} = \frac{A}{x+2} + \frac{Bx + C}{(x^2+1)}$$

Therefore, $$x^2 + x + 1 = A\,(x^2 + 1) + (Bx + C)\,(x + 2)$$

Equating the coefficients of x^2, x and of constant term of both sides, we get A + B =1, 2B + C = 1 and A + 2C = 1. Solving these equations, we get $A=\frac{3}{5}, B=\frac{2}{5}$ and $C=\frac{1}{5}$

Thus, the integrand is given by

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{3}{5(x+2)}+\frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1} = \frac{3}{5(x+2)}+\frac{1}{5}\left(\frac{2x+1}{x^2+1}\right)$$

Therefore, $$\int\frac{x^2+x+1}{(x^2+1)(x+2)}dx = \frac{3}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{2x}{x^2+1}dx+\frac{1}{5}\int\frac{1}{x^2+1}dx$$

$$= \frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$$

EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

1. $\frac{x}{(x+1)(x+2)}$ **2.** $\frac{1}{x^2-9}$ **3.** $\frac{3x-1}{(x-1)(x-2)(x-3)}$

4. $\frac{x}{(x-1)(x-2)(x-3)}$ **5.** $\frac{2x}{x^2+3x+2}$ **6.** $\frac{1-x^2}{x(1-2x)}$

7. $\frac{x}{(x^2+1)(x-1)}$ **8.** $\frac{x}{(x-1)^2(x+2)}$ **9.** $\frac{3x+5}{x^3-x^2-x+1}$

10. $\frac{2x-3}{(x^2-1)(2x+3)}$ **11.** $\frac{5x}{(x+1)(x^2-4)}$ **12.** $\frac{x^3+x+1}{x^2-1}$

13. $\frac{2}{(1-x)(1+x^2)}$ **14.** $\frac{3x-1}{(x+2)^2}$ **15.** $\frac{1}{x^4-1}$

16. $\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n=t$]

17. $\frac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint : Put $\sin x = t$]

18. $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ **19.** $\dfrac{2x}{(x^2+1)(x^2+3)}$ **20.** $\dfrac{1}{x(x^4-1)}$

21. $\dfrac{1}{(e^x-1)}$ [Hint : Put $e^x = t$]

Choose the correct answer in each of the Exercises 22 and 23.

22. $\displaystyle\int \frac{x\,dx}{(x-1)(x-2)}$ equals

(A) $\log\left|\dfrac{(x-1)^2}{x-2}\right|+C$ (B) $\log\left|\dfrac{(x-2)^2}{x-1}\right|+C$

(C) $\log\left|\left(\dfrac{x-1}{x-2}\right)^2\right|+C$ (D) $\log|(x-1)(x-2)|+C$

23. $\displaystyle\int \frac{dx}{x(x^2+1)}$ equals

(A) $\log|x|-\dfrac{1}{2}\log(x^2+1)+C$ (B) $\log|x|+\dfrac{1}{2}\log(x^2+1)+C$

(C) $-\log|x|+\dfrac{1}{2}\log(x^2+1)+C$ (D) $\dfrac{1}{2}\log|x|+\log(x^2+1)+C$

7.6 Integration by Parts

In this section, we describe one more method of integration, that is found quite useful in integrating products of functions.

If u and v are any two differentiable functions of a single variable x (say). Then, by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx}+v\frac{du}{dx}$$

Integrating both sides, we get

$$uv = \int u\frac{dv}{dx}\,dx+\int v\frac{du}{dx}\,dx$$

or
$$\int u\frac{dv}{dx}\,dx = uv-\int v\frac{du}{dx}\,dx \qquad \dots (1)$$

Let
$$u = f(x) \text{ and } \frac{dv}{dx} = g(x). \text{ Then}$$

$$\frac{du}{dx} = f'(x) \text{ and } v = \int g(x)\,dx$$

Therefore, expression (1) can be rewritten as

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [\int g(x)\, dx]\, f'(x)\, dx$$

i.e.,
$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [f'(x) \int g(x)\, dx]\, dx$$

If we take f as the first function and g as the second function, then this formula may be stated as follows:

"The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Example 17 Find $\int x \cos x\, dx$

Solution Put $f(x) = x$ (first function) and $g(x) = \cos x$ (second function). Then, integration by parts gives

$$\int x \cos x\, dx = x \int \cos x\, dx - \int [\frac{d}{dx}(x) \int \cos x\, dx]\, dx$$

$$= x \sin x - \int \sin x\, dx = x \sin x + \cos x + C$$

Suppose, we take $f(x) = \cos x$ and $g(x) = x$. Then

$$\int x \cos x\, dx = \cos x \int x\, dx - \int [\frac{d}{dx}(\cos x) \int x\, dx]\, dx$$

$$= (\cos x)\frac{x^2}{2} + \int \sin x\, \frac{x^2}{2}\, dx$$

Thus, it shows that the integral $\int x \cos x\, dx$ is reduced to the comparatively more complicated integral having more power of x. Therefore, the proper choice of the first function and the second function is significant.

Remarks

(i) It is worth mentioning that integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for $\int \sqrt{x} \sin x\, dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x}\, \sin x$.

(ii) Observe that while finding the integral of the second function, we did not add any constant of integration. If we write the integral of the second function $\cos x$

as $\sin x + k$, where k is any constant, then

$$\int x \cos x\, dx = x(\sin x + k) - \int (\sin x + k)\, dx$$
$$= x(\sin x + k) - \int (\sin x\, dx - \int k\, dx$$
$$= x(\sin x + k) - \cos x - kx + C = x \sin x + \cos x + C$$

This shows that adding a constant to the integral of the second function is superfluous so far as the final result is concerned while applying the method of integration by parts.

(iii) Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, in cases where other function is inverse trigonometric function or logarithmic function, then we take them as first function.

Example 18 Find $\int \log x\, dx$

Solution To start with, we are unable to guess a function whose derivative is $\log x$. We take $\log x$ as the first function and the constant function 1 as the second function. Then, the integral of the second function is x.

Hence,
$$\int (\log x . 1)\, dx = \log x \int 1\, dx - \int [\frac{d}{dx}(\log x) \int 1\, dx]\, dx$$
$$= (\log x) \cdot x - \int \frac{1}{x} x\, dx = x \log x - x + C.$$

Example 19 Find $\int x\, e^x dx$

Solution Take first function as x and second function as e^x. The integral of the second function is e^x.

Therefore,
$$\int x\, e^x dx = x\, e^x - \int 1 \cdot e^x dx = xe^x - e^x + C.$$

Example 20 Find $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\, dx$

Solution Let first function be $\sin^{-1} x$ and second function be $\frac{x}{\sqrt{1-x^2}}$.

First we find the integral of the second function, i.e., $\int \frac{x\, dx}{\sqrt{1-x^2}}$.

Put $t = 1 - x^2$. Then $dt = -2x\, dx$

Therefore, $$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

Hence, $$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = (\sin^{-1}x)\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}(-\sqrt{1-x^2})\,dx$$
$$= -\sqrt{1-x^2}\,\sin^{-1}x + x + C = x - \sqrt{1-x^2}\,\sin^{-1}x + C$$

Alternatively, this integral can also be worked out by making substitution $\sin^{-1}x = \theta$ and then integrating by parts.

Example 21 Find $\int e^x \sin x\,dx$

Solution Take e^x as the first function and $\sin x$ as second function. Then, integrating by parts, we have

$$I = \int e^x \sin x\,dx = e^x(-\cos x) + \int e^x \cos x\,dx$$
$$= -e^x \cos x + I_1 \text{ (say)} \qquad \dots (1)$$

Taking e^x and $\cos x$ as the first and second functions, respectively, in I_1, we get

$$I_1 = e^x \sin x - \int e^x \sin x\,dx$$

Substituting the value of I_1 in (1), we get

$$I = -e^x \cos x + e^x \sin x - I \quad \text{or} \quad 2I = e^x(\sin x - \cos x)$$

Hence, $$I = \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Alternatively, above integral can also be determined by taking $\sin x$ as the first function and e^x the second function.

7.6.1 *Integral of the type* $\int e^x [f(x) + f'(x)]\,dx$

We have $$I = \int e^x [f(x) + f'(x)]\,dx = \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$
$$= I_1 + \int e^x f'(x)\,dx, \text{ where } I_1 = \int e^x f(x)\,dx \qquad \dots (1)$$

Taking $f(x)$ and e^x as the first function and second function, respectively, in I_1 and integrating it by parts, we have $I_1 = f(x)\,e^x - \int f'(x)\,e^x dx + C$

Substituting I_1 in (1), we get

$$I = e^x f(x) - \int f'(x)\,e^x dx + \int e^x f'(x)\,dx + C = e^x f(x) + C$$

Thus, $$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$

Example 22 Find (i) $\int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx$ (ii) $\int \frac{(x^2+1)\, e^x}{(x+1)^2}\, dx$

Solution

(i) We have $I = \int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx$

Consider $f(x) = \tan^{-1} x$, then $f'(x) = \frac{1}{1+x^2}$

Thus, the given integrand is of the form $e^x [f(x) + f'(x)]$.

Therefore, $I = \int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx = e^x \tan^{-1} x + C$

(ii) We have $I = \int \frac{(x^2+1)\, e^x}{(x+1)^2}\, dx = \int e^x [\frac{x^2 - 1 + 1 + 1)}{(x+1)^2}]\, dx$

$$= \int e^x [\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}]\, dx = \int e^x [\frac{x-1}{x+1} + \frac{2}{(x+1)^2}]\, dx$$

Consider $f(x) = \frac{x-1}{x+1}$, then $f'(x) = \frac{2}{(x+1)^2}$

Thus, the given integrand is of the form $e^x [f(x) + f'(x)]$.

Therefore, $$\int \frac{x^2+1}{(x+1)^2} e^x\, dx = \frac{x-1}{x+1} e^x + C$$

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

1. $x \sin x$ **2.** $x \sin 3x$ **3.** $x^2 e^x$ **4.** $x \log x$

5. $x \log 2x$ **6.** $x^2 \log x$ **7.** $x \sin^{-1} x$ **8.** $x \tan^{-1} x$

9. $x \cos^{-1} x$ **10.** $(\sin^{-1} x)^2$ **11.** $\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$ **12.** $x \sec^2 x$

13. $\tan^{-1} x$ **14.** $x (\log x)^2$ **15.** $(x^2 + 1) \log x$

16. $e^x(\sin x + \cos x)$ **17.** $\dfrac{x\,e^x}{(1+x)^2}$ **18.** $e^x\left(\dfrac{1+\sin x}{1+\cos x}\right)$

19. $e^x\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$ **20.** $\dfrac{(x-3)\,e^x}{(x-1)^3}$ **21.** $e^{2x}\sin x$

22. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3}\,dx$ equals

(A) $\dfrac{1}{3}e^{x^3}+\mathrm{C}$ (B) $\dfrac{1}{3}e^{x^2}+\mathrm{C}$

(C) $\dfrac{1}{2}e^{x^3}+\mathrm{C}$ (D) $\dfrac{1}{2}e^{x^2}+\mathrm{C}$

24. $\int e^x \sec x\,(1+\tan x)\,dx$ equals

(A) $e^x \cos x + \mathrm{C}$ (B) $e^x \sec x + \mathrm{C}$

(C) $e^x \sin x + \mathrm{C}$ (D) $e^x \tan x + \mathrm{C}$

7.6.2 *Integrals of some more types*

Here, we discuss some special types of standard integrals based on the technique of integration by parts :

(i) $\int\sqrt{x^2-a^2}\,dx$ (ii) $\int\sqrt{x^2+a^2}\,dx$ (iii) $\int\sqrt{a^2-x^2}\,dx$

(i) Let $\mathrm{I}=\int\sqrt{x^2-a^2}\,dx$

Taking constant function 1 as the second function and integrating by parts, we have

$$\mathrm{I} = x\sqrt{x^2-a^2} - \int\frac{1}{2}\frac{2x}{\sqrt{x^2-a^2}}\,x\,dx$$

$$= x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\, dx - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - \mathrm{I} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

or $$2\mathrm{I} = x\sqrt{x^2-a^2} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

or $$\mathbf{I = \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left| x+\sqrt{x^2-a^2}\right| + C}$$

Similarly, integrating other two integrals by parts, taking constant function 1 as the second function, we get

(ii) $$\mathbf{\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\,x\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left| x+\sqrt{x^2+a^2}\right| + C}$$

(iii) $$\mathbf{\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\,x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C}$$

Alternatively, integrals (i), (ii) and (iii) can also be found by making trigonometric substitution $x = a\sec\theta$ in (i), $x = a\tan\theta$ in (ii) and $x = a\sin\theta$ in (iii) respectively.

Example 23 Find $\int \sqrt{x^2+2x+5}\, dx$

Solution Note that

$$\int \sqrt{x^2+2x+5}\, dx = \int \sqrt{(x+1)^2+4}\, dx$$

Put $x + 1 = y$, so that $dx = dy$. Then

$$\int \sqrt{x^2+2x+5}\, dx = \int \sqrt{y^2+2^2}\, dy$$

$$= \frac{1}{2}y\sqrt{y^2+4} + \frac{4}{2}\log\left| y+\sqrt{y^2+4}\right| + C \qquad \text{[using 7.6.2 (ii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log\left| x+1+\sqrt{x^2+2x+5}\right| + C$$

Example 24 Find $\int \sqrt{3-2x-x^2}\, dx$

Solution Note that $\int \sqrt{3-2x-x^2}\, dx = \int \sqrt{4-(x+1)^2}\, dx$

Put $x + 1 = y$ so that $dx = dy$.

Thus
$$\int\sqrt{3-2x-x^2}\,dx = \int\sqrt{4-y^2}\,dy$$
$$= \frac{1}{2}y\sqrt{4-y^2}+\frac{4}{2}\sin^{-1}\frac{y}{2}+C \qquad \text{[using 7.6.2 (iii)]}$$
$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2}+2\sin^{-1}\left(\frac{x+1}{2}\right)+C$$

EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

1. $\sqrt{4-x^2}$ **2.** $\sqrt{1-4x^2}$ **3.** $\sqrt{x^2+4x+6}$

4. $\sqrt{x^2+4x+1}$ **5.** $\sqrt{1-4x-x^2}$ **6.** $\sqrt{x^2+4x-5}$

7. $\sqrt{1+3x-x^2}$ **8.** $\sqrt{x^2+3x}$ **9.** $\sqrt{1+\frac{x^2}{9}}$

Choose the correct answer in Exercises 10 to 11.

10. $\int\sqrt{1+x^2}\,dx$ is equal to

(A) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right|+C$

(B) $\frac{2}{3}(1+x^2)^{\frac{3}{2}}+C$ (C) $\frac{2}{3}x(1+x^2)^{\frac{3}{2}}+C$

(D) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

11. $\int\sqrt{x^2-8x+7}\,dx$ is equal to

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

7.7 Definite Integral

In the previous sections, we have studied about the indefinite integrals and discussed few methods of finding them including integrals of some special functions. In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)\,dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$. Here, we shall consider these two cases separately as discussed below:

7.7.1 *Definite integral as the limit of a sum*

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x = a$ and $x = b$ (Fig 7.2).

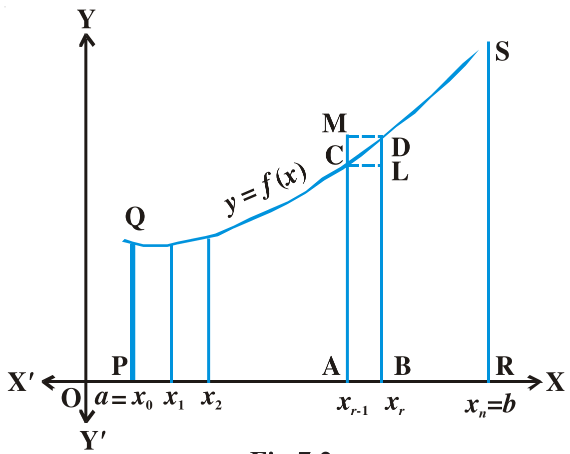

Fig 7.2

Divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1]$, $[x_1, x_2]$,...., $[x_{r-1}, x_r]$, ..., $[x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... , $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \dfrac{b-a}{h}$. We note that as $n \to \infty$, $h \to 0$.

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots, n$.

From Fig 7.2, we have

area of the rectangle (ABLC) < area of the region (ABDCA) < area of the rectangle (ABDM) ... (1)

Evidently as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h\,[f(x_0) + \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \qquad \ldots (2)$$

and
$$S_n = h\,[f(x_1) + f(x_2) + \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \qquad \ldots (3)$$

Here, s_n and S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$, respectively.

In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region PRSQP} < S_n \qquad \ldots (4)$$

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region PRSQP} = \int_a^b f(x)dx \qquad \ldots (5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h]$$

or
$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h] \qquad \ldots (6)$$

where
$$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression (6) is known as the definition of definite integral as the *limit of sum.*

Remark The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)\,dt$ or $\int_a^b f(u)\,du$ instead of $\int_a^b f(x)\,dx$. Hence, the variable of integration is called a *dummy variable.*

Example 25 Find $\int_0^2 (x^2+1)\,dx$ as the limit of a sum.

Solution By definition

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+...+f(a+(n-1)\,h],$$

where, $$h = \frac{b-a}{n}$$

In this example, $a = 0$, $b = 2$, $f(x) = x^2 + 1$, $h = \frac{2-0}{n} = \frac{2}{n}$

Therefore,

$$\int_0^2 (x^2+1)\,dx = 2\lim_{n\to\infty}\frac{1}{n}[f(0)+f(\frac{2}{n})+f(\frac{4}{n})+...+f(\frac{2\,(n-1)}{n})]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[1+(\frac{2^2}{n^2}+1)+(\frac{4^2}{n^2}+1)+...+\left(\frac{(2n-2)^2}{n^2}+1\right)]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[\underbrace{(1+1+...+1)}_{n\text{-terms}}+\frac{1}{n^2}(2^2+4^2+...+(2n-2)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2^2}{n^2}(1^2+2^2+...+(n-1)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{4}{n^2}\frac{(n-1)\,n\,(2n-1)}{6}]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2}{3}\frac{(n-1)\;(2n-1)}{n}]$$

$$= 2\lim_{n\to\infty}[1+\frac{2}{3}(1-\frac{1}{n})\;(2-\frac{1}{n})] = 2\;[1+\frac{4}{3}] = \frac{14}{3}$$

Example 26 Evaluate $\int_0^2 e^x\,dx$ as the limit of a sum.

Solution By definition

$$\int_0^2 e^x\,dx = (2-0)\lim_{n\to\infty}\frac{1}{n}\left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}}\right]$$

Using the sum to n terms of a G.P., where $a = 1$, $r = e^{\frac{2}{n}}$, we have

$$\int_0^2 e^x\,dx = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{2\,(e^2-1)}{\lim\limits_{n\to\infty}\left[\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right]\cdot 2} = e^2 - 1 \qquad \left[\text{using } \lim_{h\to 0}\frac{(e^h-1)}{h} = 1\right]$$

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x\,dx$
2. $\int_0^5 (x+1)\,dx$
3. $\int_2^3 x^2\,dx$
4. $\int_1^4 (x^2 - x)\,dx$
5. $\int_{-1}^1 e^x\,dx$
6. $\int_0^4 (x + e^{2x})\,dx$

7.8 Fundamental Theorem of Calculus

7.8.1 *Area function*

We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and x-axis. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the area of the light shaded region

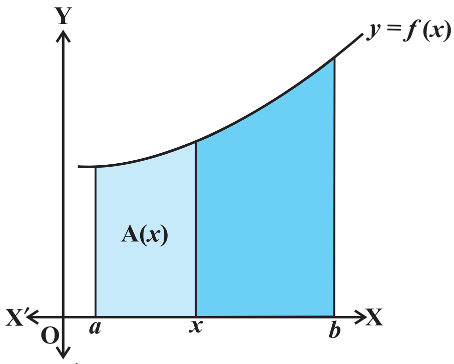

Fig 7.3

in Fig 7.3 [Here it is assumed that $f(x) > 0$ for $x \in [a, b]$, the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of x.

In other words, the area of this shaded region is a function of x. We denote this function of x by A(x). We call the function A(x) as *Area function* and is given by

$$\mathbf{A}(x) = \int_a^x f(x)\,dx \qquad \text{... (1)}$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

7.8.2 *First fundamental theorem of integral calculus*

Theorem 1 Let f be a continuous function on the closed interval $[a, b]$ and let A (x) be the area function. Then $\mathbf{A}'(x) = f(x)$, for all $x \in [a, b]$.

7.8.3 *Second fundamental theorem of integral calculus*

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 2 Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$.

Remarks

(i) In words, the Theorem 2 tells us that $\int_a^b f(x)\,dx$ = (value of the anti derivative F of f at the upper limit b – value of the same anti derivative at the lower limit a).

(ii) This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

(iii) The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.

(iv) In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in $[a, b]$.

For instance, the consideration of definite integral $\int_{-2}^3 x(x^2-1)^{\frac{1}{2}}\,dx$ is erroneous since the function f expressed by $f(x) = x(x^2-1)^{\frac{1}{2}}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

Steps for calculating $\int_a^b f(x)\,dx$.

(i) Find the indefinite integral $\int f(x)\,dx$. Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get

$$\int_a^b f(x)\,dx = [F(x) + C]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a).$$

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate F(b) – F(a) = $[F(x)]_a^b$, which is the value of $\int_a^b f(x)\,dx$.

We now consider some examples

Example 27 Evaluate the following integrals:

(i) $\int_2^3 x^2\,dx$ (ii) $\int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx$

(iii) $\int_1^2 \frac{x\,dx}{(x+1)(x+2)}$ (iv) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\,dt$

Solution

(i) Let $I = \int_2^3 x^2\,dx$. Since $\int x^2\,dx = \frac{x^3}{3} = F(x)$,

Therefore, by the second fundamental theorem, we get

$$I = F(3) - F(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

(ii) Let $I = \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx$. We first find the anti derivative of the integrand.

Put $30 - x^{\frac{3}{2}} = t$. Then $-\frac{3}{2}\sqrt{x}\,dx = dt$ or $\sqrt{x}\,dx = -\frac{2}{3}dt$

Thus, $$\int \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx = -\frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right] = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right] = F(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$\mathrm{I} = \mathrm{F}(9) - \mathrm{F}(4) = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30-27)} - \frac{1}{30-8}\right] = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right] = \frac{19}{99}$$

(iii) Let $\mathrm{I} = \int_1^2 \frac{x\,dx}{(x+1)(x+2)}$

Using partial fraction, we get $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

So $$\int \frac{x\,dx}{(x+1)(x+2)} = -\log|x+1| + 2\log|x+2| = \mathrm{F}(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$\mathrm{I} = \mathrm{F}(2) - \mathrm{F}(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3]$$

$$= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right)$$

(iv) Let $\mathrm{I} = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\, dt$. Consider $\int \sin^3 2t \cos 2t\, dt$

Put $\sin 2t = u$ so that $2\cos 2t\, dt = du$ or $\cos 2t\, dt = \frac{1}{2}\, du$

So $$\int \sin^3 2t \cos 2t\, dt = \frac{1}{2}\int u^3 du$$

$$= \frac{1}{8}[u^4] = \frac{1}{8}\sin^4 2t = \mathrm{F}(t) \text{ say}$$

Therefore, by the second fundamental theorem of integral calculus

$$\mathrm{I} = \mathrm{F}\left(\frac{\pi}{4}\right) - \mathrm{F}(0) = \frac{1}{8}\left[\sin^4 \frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

1. $\int_{-1}^{1}(x+1)\,dx$ **2.** $\int_{2}^{3}\frac{1}{x}\,dx$ **3.** $\int_{1}^{2}(4x^3-5x^2+6x+9)\,dx$

4. $\int_{0}^{\frac{\pi}{4}}\sin 2x\,dx$ **5.** $\int_{0}^{\frac{\pi}{2}}\cos 2x\,dx$ **6.** $\int_{4}^{5}e^x\,dx$ **7.** $\int_{0}^{\frac{\pi}{4}}\tan x\,dx$

8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\operatorname{cosec} x\,dx$ **9.** $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$ **10.** $\int_{0}^{1}\frac{dx}{1+x^2}$ **11.** $\int_{2}^{3}\frac{dx}{x^2-1}$

12. $\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx$ **13.** $\int_{2}^{3}\frac{x\,dx}{x^2+1}$ **14.** $\int_{0}^{1}\frac{2x+3}{5x^2+1}\,dx$ **15.** $\int_{0}^{1}x\,e^{x^2}\,dx$

16. $\int_{1}^{2}\frac{5x^2}{x^2+4x+3}$ **17.** $\int_{0}^{\frac{\pi}{4}}(2\sec^2 x+x^3+2)\,dx$ **18.** $\int_{0}^{\pi}(\sin^2\frac{x}{2}-\cos^2\frac{x}{2})\,dx$

19. $\int_{0}^{2}\frac{6x+3}{x^2+4}\,dx$ **20.** $\int_{0}^{1}(x\,e^x+\sin\frac{\pi x}{4})\,dx$

Choose the correct answer in Exercises 21 and 22.

21. $\int_{1}^{\sqrt{3}}\frac{dx}{1+x^2}$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

22. $\int_{0}^{\frac{2}{3}}\frac{dx}{4+9x^2}$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

7.9 Evaluation of Definite Integrals by Substitution

In the previous sections, we have discussed several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution.

To evaluate $\int_a^b f(x)\,dx$, by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute, $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form.
2. Integrate the new integrand with respect to the new variable without mentioning the constant of integration.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

Note In order to quicken this method, we can proceed as follows: After performing steps 1, and 2, there is no need of step 3. Here, the integral will be kept in the new variable itself, and the limits of the integral will accordingly be changed, so that we can perform the last step.

Let us illustrate this by examples.

Example 28 Evaluate $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$.

Solution Put $t = x^5 + 1$, then $dt = 5x^4\,dx$.

Therefore, $$\int 5x^4\sqrt{x^5+1}\,dx = \int \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$$

Hence, $$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{2}{3}\left[(1^5+1)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Alternatively, first we transform the integral and then evaluate the transformed integral with new limits.

Let $t = x^5 + 1$. Then $dt = 5x^4\, dx$.

Note that, when $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

Thus, as x varies from -1 to 1, t varies from 0 to 2

Therefore $$\int_{-1}^{1} 5x^4 \sqrt{x^5+1}\, dx = \int_0^2 \sqrt{t}\, dt$$

$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Example 29 Evaluate $\displaystyle\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$

Solution Let $t = \tan^{-1}x$, then $dt = \dfrac{1}{1+x^2}\, dx$. The new limits are, when $x = 0, t = 0$ and when $x = 1$, $t = \dfrac{\pi}{4}$. Thus, as x varies from 0 to 1, t varies from 0 to $\dfrac{\pi}{4}$.

Therefore $$\int_0^1 \frac{\tan^{-1}x}{1+x^2}\, dx = \int_0^{\frac{\pi}{4}} t\, dt \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

1. $\displaystyle\int_0^1 \frac{x}{x^2+1}\, dx$ **2.** $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\, \cos^5\phi\, d\phi$ **3.** $\displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

4. $\displaystyle\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$) **5.** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$

6. $\displaystyle\int_0^2 \frac{dx}{x+4-x^2}$ **7.** $\displaystyle\int_{-1}^1 \frac{dx}{x^2+2x+5}$ **8.** $\displaystyle\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\displaystyle\int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\, dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

10. If $f(x) = \displaystyle\int_0^x t \sin t\, dt$, then $f'(x)$ is

(A) $\cos x + x \sin x$ (B) $x \sin x$

(C) $x \cos x$ (D) $\sin x + x \cos x$

7.10 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

P_0 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$

P_2 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

P_3 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P_4 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(Note that P_4 is a particular case of P_3)

P_5 : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P_6 : $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$ and

0 if $f(2a-x) = -f(x)$

P_7 : (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function, i.e., if $f(-x) = f(x)$.

(ii) $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.

We give the proofs of these properties one by one.

Proof of P_0 It follows directly by making the substitution $x = t$.

Proof of P_1 Let F be anti derivative of f. Then, by the second fundamental theorem of calculus, we have $\int_a^b f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\,dx$

Here, we observe that, if $a = b$, then $\int_a^a f(x)\,dx = 0$.

Proof of P_2 Let F be anti derivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{... (1)}$$

$$\int_a^c f(x)\,dx = F(c) - F(a) \quad \text{... (2)}$$

and

$$\int_c^b f(x)\,dx = F(b) - F(c) \quad \text{... (3)}$$

Adding (2) and (3), we get $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) = \int_a^b f(x)\,dx$

This proves the property P_2.

Proof of P_3 Let $t = a + b - x$. Then $dt = -\,dx$. When $x = a$, $t = b$ and when $x = b$, $t = a$. Therefore

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \quad (\text{by } P_1)$$

$$= \int_a^b f(a+b-x)\,dx \text{ by } P_0$$

Proof of P_4 Put $t = a - x$. Then $dt = -\,dx$. When $x = 0$, $t = a$ and when $x = a$, $t = 0$. Now proceed as in P_3.

Proof of P_5 Using P_2, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$.

Let $t = 2a - x$ in the second integral on the right hand side. Then $dt = -\,dx$. When $x = a$, $t = a$ and when $x = 2a$, $t = 0$. Also $x = 2a - t$.

Therefore, the second integral becomes

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

Hence $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Proof of P_6 Using P_5, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ ... (1)

Now, if $f(2a - x) = f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx,$$

and if $f(2a - x) = -f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

Proof of P_7 Using P_2, we have

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx. \text{ Then}$$

Let $t = -x$ in the first integral on the right hand side. $dt = -\,dx$. When $x = -a$, $t = a$ and when $x = 0$, $t = 0$. Also $x = -t$.

Therefore $$\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{(by } P_0\text{)} \quad \ldots (1)$$

(i) Now, if f is an even function, then $f(-x) = f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(ii) If f is an odd function, then $f(-x) = -f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Example 30 Evaluate $\int_{-1}^{2} \left| x^3 - x \right| dx$

Solution We note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and that $x^3 - x \geq 0$ on $[1, 2]$. So by P_2 we write

$$\int_{-1}^{2} \left| x^3 - x \right| dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

Example 31 Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx$

Solution We observe that $\sin^2 x$ is an even function. Therefore, by P_7 (i), we get

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x\,dx$$